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On the mechanics of solids with a growing mass

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Abstract

A general constitutive theory of the stress-modulated growth of biomaterials is presented with a particular accent given to pseudo-elastic soft living tissues. The governing equations of the mechanics of solids with a growing mass are revisited within the framework of finite deformation continuum thermodynamics. The multiplicative decomposition of the deformation gradient into its elastic and growth parts is employed to study the growth of isotropic, transversely isotropic, and orthotropic biomaterials. An explicit representation of the growth part of the deformation gradient is given in each case, which leads to an effective incremental formulation in the analysis of the stress-modulated growth process. The rectangular components of the instantaneous elastic moduli tensor are derived corresponding to selected forms of the elastic strain energy function. Physically appealing structures of the stress-dependent evolution equations for the growth induced stretch ratios are proposed.
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1. Introduction

The analysis of stress-modulated growth of living tissues, bones, and other biomaterials has been an important research topic in biomechanics during past several decades. Early work includes a study of the relationship between the mechanical loads and uniform growth by Hsu (1968), and a study of the mass deposition and resorption processes in a living bone (hard tissue) by Cowin and Hegedus (1976a,b). The latter work provided a set of governing equations for the so-called adaptive elasticity theory, in which an elastic material adopts its structure to applied loading (see also Cowin, 1990). Fundamental contributions were further made by Skalak et al. (1982) in their analytical description of the volumetrically distributed mass growth, and the mass growth by deposition or resorption on a surface. The origin and the role of residual stresses in biological tissues have been examined both analytically and experimentally by many researchers. The review papers by Humphrey (1995) and Taber (1995) contain an extensive list of related

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references. In contrast to hard tissues which undergo only small deformations, soft tissues such as blood vessels and tendons can experience large deformations. An important contribution to the general study of finite volumetric growth in soft elastic tissues was made by Rodrigez et al. (1994). They introduced and employed the multiplicative decomposition of the total deformation gradient into its elastic and growth parts. Subsequent work includes the contributions by Taber and Eggers (1996), Taber and Perucchio (2000), Chen and Hoger (2000), and Hoger et al. (submitted). The use of this decomposition in the constitutive theory of the stress-modulated growth of soft tissues is further pursued in this paper.

In Section 2 we present the spatial and material form of the continuity equation for the continuum with a growing mass, and derive some general relationships between the quantities reckoned per unit initial and current mass. These results are used in Section 3 to construct a simple proof of the corresponding Reynolds transport theorem. The momentum principles and the rate-type equations of motion are summarized in Section 4. Various forms of the energy and entropy equations are presented in Sections 5 and 6. By employing the framework of the thermodynamics with internal state variables and the appropriate thermodynamic potentials, we deduce in Section 7 the general structure of the constitutive equations for the continuum with a growing mass. The analysis is further pursued by introducing the multiplicative decomposition of the deformation gradient into its elastic and growth parts. Kinematic and kinetic aspects of this decomposition are discussed in Section 8. The relationships between the mass densities and their time rates relative to the stressed and unstressed material configurations are given in Section 9. The isothermal elastic stress response is considered in Section 10, and the partition of the rate of deformation into elastic and growth parts in Section 11. A detailed analysis of the stress-modulated mass growth of an isotropic material is presented in Section 12. An elastic strain energy function appropriate for soft tissues is used, and a stress-dependent evolution equation for the growth stretch ratio is proposed. Transversely isotropic materials with transversely isotropic mass growth are considered in Section 13. The components of instantaneous elastic moduli tensor are derived corresponding to selected form of the strain energy function. An explicit representation for the growth part of the deformation gradient is constructed. Evolution equations for the longitudinal and transverse growth stretch ratios are then proposed. Section 14 is devoted to orthotropic materials with orthotropic mass growth, while Section 15 deals with orthotropic materials whose structure allows a transversely isotropic mass growth. The appropriate representations of the elastic strain energy functions are considered, and the corresponding components of instantaneous elastic moduli tensors are derived. The growth parts of the deformation gradient are specified, and the stress-dependent evolution equations for the principal growth stretch ratios are proposed. The concluding remarks with suggestions for further research are summarized in Section 16.

2. Continuity equation

Let r_g be a time rate of the mass growth per unit current volume. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}m) = r_g \,\mathrm{d}V, \tag{2.1}$$

where $\mathrm{d}/\mathrm{d}t$ stands for the material time derivative. For $r_g > 0$ the mass growth occurs, and for $r_g < 0$ the mass resorption takes place. If $\rho = \mathrm{d}m/\mathrm{d}V$ is the mass density, we obtain from Eq. (2.1)

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}\mathrm{d}V + \rho\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}V) = r_g \,\mathrm{d}V. \tag{2.2}$$

Since the volume rate is proportional to the divergence of velocity field,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}V) = (\nabla\cdot\mathbf{v})\,\mathrm{d}V, \tag{2.3}$$

the substitution into Eq. (2.2) gives the continuity equation for the continuum with a growing mass

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\nabla\cdot\mathbf{v} = r_{\mathrm{g}}. \tag{2.4}$$

This equation is commonly derived in the literature by considering a control volume fixed in space, the mass influx across its boundary, and the mass creation from internal sources within the control volume (e.g., Hsu, 1968; Cowin and Hegedus, 1976a,b). We have chosen the above derivation instead as simpler and more straightforward.

The integral form of the continuity equation follows from the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}m = \int_V \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}m), \tag{2.5}$$

where V is the current volume of the considered material sample. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V = \int_V r_{\mathrm{g}}\,\mathrm{d}V. \tag{2.6}$$

Initially, before the deformation and mass growth, $\rho = \rho_0$ (initial mass density).

For isochoric (volume preserving) deformation and growth

$$\nabla\cdot\mathbf{v} = 0, \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = r_{\mathrm{g}}, \tag{2.7}$$

while for incompressible materials

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0, \quad \nabla\cdot\mathbf{v} = \frac{1}{\rho}r_{\mathrm{g}}. \tag{2.8}$$

2.1. Material form of continuity equation

If $\mathbf{F}$ is a deformation gradient, and $J = \det\mathbf{F}$ is its determinant, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho\,\mathrm{d}V) = r_{\mathrm{g}}\,\mathrm{d}V \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\rho J\,\mathrm{d}V^0) = r_{\mathrm{g}}J\,\mathrm{d}V^0, \tag{2.9}$$

where $\mathrm{d}V^0$ is the initial volume element, and $\mathrm{d}V = J\,\mathrm{d}V^0$. We assume that material points are everywhere dense during the mass growth, so that in any small neighborhood around the particle there are always points that existed before the growth (Chen and Hoger, 2000). Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho J) = r_{\mathrm{g}}J. \tag{2.10}$$

Furthermore, introduce a time rate of the mass growth per unit initial volume r_{g}^0, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}m) = r_{\mathrm{g}}\,\mathrm{d}V = r_{\mathrm{g}}^0\,\mathrm{d}V^0. \tag{2.11}$$

It follows that the rate of mass growth per unit initial and current volume are related by

$$r_{\mathrm{g}}^0 = r_{\mathrm{g}}J. \tag{2.12}$$

Consequently, we can rewrite Eq. (2.10) as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho J) = r_{\mathrm{g}}^0. \tag{2.13}$$

Upon the time integration this gives

$$\rho J = \rho^0 + \int_0^t r_{\rm g}^0 \, {\rm d}\tau. \tag{2.14}$$

The physical interpretation of the integral on the right-hand side is available from

$$\int_0^t r_{\rm g}^0 \, {\rm d}\tau = \frac{({\rm d}m)^t - ({\rm d}m)^0}{{\rm d}V^0}. \tag{2.15}$$

In Eq. (2.14) we have

$$\rho = \rho[\mathbf{x}(\mathbf{X}, t)], \quad J = J[\mathbf{x}(\mathbf{X}, t)], \quad r_{\rm g}^0 = r_{\rm g}^0[\mathbf{x}(\mathbf{X}, \tau)], \tag{2.16}$$

where $\mathbf{x}$ is the current position of the material point initially at $\mathbf{X}$, and $0 \leqslant \tau \leqslant t$. Eq. (2.14) represents a material form of the continuity equation for the continuum with a growing mass. It will be convenient in the sequel to designate the quantity ρJ by the symbol $\rho_{\rm g}^0$, i.e.,

$$\rho_{\rm g}^0 = \frac{({\rm d}m)^t}{{\rm d}V^0} = \rho J, \quad \frac{{\rm d}\rho_{\rm g}^0}{{\rm d}t} = r_{\rm g}^0. \tag{2.17}$$

For a continuum without the mass growth (${\rm d}m = {\rm const.}$), $\rho_{\rm g}^0 = \rho^0$.

If the volume remains preserved throughout the course of deformation and growth (densification), $J = 1$ and

$$\rho_{\rm g}^0 = \rho, \quad r_{\rm g} = r_{\rm g}^0, \tag{2.18}$$

$$\rho = \rho^0 + \int_0^t r_{\rm g}^0 \, {\rm d}\tau. \tag{2.19}$$

For an incompressible material that remains incompressible during the mass growth ($\rho = \rho^0$), Eq. (2.14) gives

$$J = 1 + \frac{1}{\rho^0} \int_0^t r_{\rm g}^0 \, {\rm d}\tau. \tag{2.20}$$

2.2. Relationships between the quantities per unit initial and current mass

Consider a quantity A per unit current mass and the corresponding quantity A^0 per unit initial mass, defined such that $A({\rm d}m)^t = A^0({\rm d}m)^0$. It follows that

$$\rho A \, {\rm d}V = \rho^0 A^0 \, {\rm d}V^0, \tag{2.21}$$

i.e.,

$$\rho^0 A^0 = \rho_{\rm g}^0 A. \tag{2.22}$$

Thus,

$$A^0 = \left(1 + \frac{1}{\rho^0} \int_0^t r_{\rm g}^0 \, {\rm d}\tau\right) A. \tag{2.23}$$

By differentiating Eq. (2.22) we also have

$$\rho^0 \frac{{\rm d}A^0}{{\rm d}t} = J\left(\rho \frac{{\rm d}A}{{\rm d}t} + r_{\rm g} A\right), \tag{2.24}$$

$$\rho_g^0 \frac{dA}{dt} = \rho^0 \left(\frac{dA^0}{dt} - \frac{r_g^0}{\rho_g^0} A^0 \right). \tag{2.25}$$

It is noted that

$$\frac{dA^0}{dt} \neq \left(\frac{dA}{dt} \right)^0, \tag{2.26}$$

where the latter quantity is defined by

$$\rho \frac{dA}{dt} \, dV = \rho^0 \left(\frac{dA}{dt} \right)^0 dV^0, \tag{2.27}$$

i.e.,

$$\rho^0 \left(\frac{dA}{dt} \right)^0 = \rho_g^0 \frac{dA}{dt}. \tag{2.28}$$

Evidently, by comparing Eqs. (2.25) and (2.28) we establish the connection

$$\left(\frac{dA}{dt} \right)^0 = \frac{dA^0}{dt} - \frac{r_g^0}{\rho_g^0} A^0. \tag{2.29}$$

3. Reynolds transport theorem

The integration of Eq. (2.21) gives

$$\frac{d}{dt} \int_V \rho A \, dV = \int_{V^0} \rho^0 \frac{dA^0}{dt} \, dV^0. \tag{3.1}$$

The substitution of Eq. (2.24) into Eq. (3.1) yields the Reynolds transport formula for the continuum with a growing mass

$$\frac{d}{dt} \int_V \rho A \, dV = \int_V \left(\rho \frac{dA}{dt} + r_g A \right) dV. \tag{3.2}$$

It is instructive to provide an alternative derivation. Consider the quantity $\mathscr{A}$ per unit current volume. Since

$$\frac{d}{dt} \int_V \mathscr{A} \, dV = \int_V \frac{d}{dt} (\mathscr{A} \, dV), \tag{3.3}$$

upon differentiation under the integral sign on the right-hand side we deduce an important formula of continuum mechanics (e.g., Williams (1985), who derives it by the control volume considerations)

$$\frac{d}{dt} \int_V \mathscr{A} \, dV = \int_V \left(\frac{d\mathscr{A}}{dt} + \mathscr{A} \nabla \cdot \mathbf{v} \right) dV. \tag{3.4}$$

This result also follows from

$$\frac{d}{dt} \int_V \mathscr{A} \, dV = \frac{d}{dt} \int_{V^0} \mathscr{A}^0 \, dV^0 = \int_{V^0} \frac{d\mathscr{A}^0}{dt} \, dV^0, \tag{3.5}$$

where $\mathscr{A}^0$ is the quantity per unit initial volume ($\mathscr{A}^0 = \mathscr{A}J$). Upon differentiation,

$$\frac{\mathrm{d}\mathscr{A}^0}{\mathrm{d}t} = J\left(\frac{\mathrm{d}\mathscr{A}}{\mathrm{d}t} + \mathscr{A}\nabla\cdot\mathbf{v}\right) \tag{3.6}$$

and substitution into Eq. (3.5) yields Eq. (3.4). The Reynolds transport formula (3.2) follows from Eq. (3.4) by writing $\mathscr{A} = \rho A$, and by using the continuity equation (2.4). See also a related analysis by Kelly (1964), and Green and Naghdi (1965).

4. The momentum principles

The first Euler's law of motion for the continuum with a growing mass can be expressed as (e.g., Cowin and Hegedus, 1976a; Klisch and Van Dyke, 2001)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{v}\,\mathrm{d}V = \int_S \mathbf{t}\,\mathrm{d}S + \int_V \rho\mathbf{b}\,\mathrm{d}V + \int_V r_\mathrm{g}\mathbf{v}\,\mathrm{d}V. \tag{4.1}$$

In addition to applied surface (**t**) and body (**b**) forces, the time rate of change of the momentum is affected by the momentum rate associated with a growing mass. This is given by the last integral on the right-hand side of Eq. (4.1). Since by the Reynolds transport theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{v}\,\mathrm{d}V = \int_V \left(\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + r_\mathrm{g}\mathbf{v}\right)\mathrm{d}V, \tag{4.2}$$

the substitution into Eq. (4.1) yields the usual differential equations of motion

$$\nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}. \tag{4.3}$$

The Cauchy stress $\boldsymbol{\sigma}$ is related to the traction vector $\mathbf{t}$ by $\mathbf{t} = \mathbf{n}\cdot\boldsymbol{\sigma}$, where $\mathbf{n}$ is the unit outward normal to the surface S bounding the volume V.

An alternative derivation proceeds by applying the momentum principle to an infinitesimal parallelepiped of volume $\mathrm{d}V$, whose sides are parallel to coordinate directions $\mathbf{e}_i$ $(i = 1,2,3)$, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\rho\,\mathrm{d}V) = \mathbf{b}\rho\,\mathrm{d}V + \frac{\partial\mathbf{t}_i}{\partial x_i}\mathrm{d}V + \mathbf{v}r_\mathrm{g}\,\mathrm{d}V. \tag{4.4}$$

The traction vector over the side with a normal $\mathbf{e}_i$ is denoted by $\mathbf{t}_i$, so that $\partial\mathbf{t}_i/\partial x_i$ multiplied by $\mathrm{d}V$ is the net force from the surface tractions on all sides of the element. Incorporating

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\rho\,\mathrm{d}V) = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\rho\,\mathrm{d}V + \mathbf{v}r_\mathrm{g}\,\mathrm{d}V \tag{4.5}$$

into Eq. (4.4), we obtain

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \rho\mathbf{b} + \frac{\partial\mathbf{t}_i}{\partial x_i}. \tag{4.6}$$

Having regard to $\mathbf{t}_i = \mathbf{e}_i\cdot\boldsymbol{\sigma}$, and

$$\frac{\partial\mathbf{t}_i}{\partial x_i} = \nabla\cdot\boldsymbol{\sigma}, \tag{4.7}$$

the substitution into Eq. (4.6) yields the equations of motion (4.3).

The material form (relative to initial configuration) of the first Euler's law of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V^0} \rho^0_\mathrm{g}\mathbf{v}\,\mathrm{d}V^0 = \int_{S^0} \mathbf{t}^0\,\mathrm{d}S^0 + \int_{V^0} \rho^0\mathbf{b}^0\,\mathrm{d}V^0 + \int_{V^0} r^0_\mathrm{g}\mathbf{v}\,\mathrm{d}V^0, \tag{4.8}$$

where

$$\rho^0\mathbf{b}^0 = \rho_g^0\mathbf{b}. \tag{4.9}$$

Since

$$\frac{d}{dt}\int_{V^0}\rho_g^0\mathbf{v}\,dV^0 = \int_{V^0}\left(\rho_g^0\frac{d\mathbf{v}}{dt} + r_g^0\mathbf{v}\right)dV^0, \tag{4.10}$$

the substitution into Eq. (4.8) yields the material form of the differential equations of motion

$$\nabla^0\cdot\mathbf{P} + \rho^0\mathbf{b}^0 = \rho_g^0\frac{d\mathbf{v}}{dt}. \tag{4.11}$$

The nominal stress $\mathbf{P}$ is related to the nominal traction $\mathbf{t}^0$ by $\mathbf{t}^0 = \mathbf{n}^0\cdot\mathbf{P}$, where $\mathbf{n}^0$ is the unit normal to the surface S^0 bounding the initial volume V^0. The well-known connections $\mathbf{t}\,dS = \mathbf{t}^0\,dS^0$ and $\mathbf{P} = \mathbf{F}^{-1}\cdot\boldsymbol{\tau}$ are recalled, where $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ is the Kirchhoff stress. The accompanying traction boundary condition is $\mathbf{n}^0\cdot\mathbf{P} = \mathbf{p}_n$, over the part of the bounding surface where the traction $\mathbf{p}_n$ is prescribed.

The integral form of the second Euler's law of motion for the continuum with a growing mass is

$$\frac{d}{dt}\int_V(\mathbf{x}\times\rho\mathbf{v})\,dV = \int_S(\mathbf{x}\times\mathbf{t})\,dS + \int_V(\mathbf{x}\times\rho\mathbf{b})\,dV + \int_V r_g(\mathbf{x}\times\mathbf{v})\,dV. \tag{4.12}$$

Since by the Reynolds transport theorem

$$\frac{d}{dt}\int_V(\mathbf{x}\times\rho\mathbf{v})\,dV = \int_V\left[\rho\frac{d}{dt}(\mathbf{x}\times\mathbf{v}) + r_g(\mathbf{x}\times\mathbf{v})\right]dV, \tag{4.13}$$

the substitution into Eq. (4.12) gives

$$\int_V\rho\frac{d}{dt}(\mathbf{x}\times\mathbf{v})\,dV = \int_S(\mathbf{x}\times\mathbf{t})\,dS + \int_V\rho(\mathbf{x}\times\mathbf{b})\,dV. \tag{4.14}$$

This is the same expression as in the mass-conserving continuum, which therefore implies the symmetry of the Cauchy stress tensor ($\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$).

4.1. Rate-type equations of motion

By differentiating Eq. (4.11) we obtain the rate-type equations of motion

$$\nabla^0\cdot\frac{d\mathbf{P}}{dt} + \rho^0\frac{d\mathbf{b}^0}{dt} = \rho_g^0\frac{d^2\mathbf{v}}{dt^2} + r_g^0\frac{d\mathbf{v}}{dt}. \tag{4.15}$$

The rate of body force is

$$\frac{d\mathbf{b}^0}{dt} = \frac{1}{\rho^0}\left(\rho_g^0\frac{d\mathbf{b}}{dt} + r_g^0\mathbf{b}\right) = \left(\frac{d\mathbf{b}}{dt}\right)^0 + \frac{r_g^0}{\rho_g^0}\mathbf{b}^0, \tag{4.16}$$

in accordance with the general recipe (2.29). The accompanying rate-type boundary condition is

$$\mathbf{n}^0\cdot\frac{d\mathbf{P}}{dt} = \frac{d\mathbf{p}_n}{dt}, \tag{4.17}$$

for the part of the bounding surface where the rate of traction is prescribed.

5. Energy equation

The rate at which the external surface and body forces are doing work on the current mass is given by the standard expression (e.g., Malvern, 1969)

$$\mathscr{P} = \int_S \mathbf{t}\cdot\mathbf{v}\,\mathrm{d}S + \int_V \rho\mathbf{b}\cdot\mathbf{v}\,\mathrm{d}V = \int_V \left[\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) + \boldsymbol{\sigma}:\mathbf{D}\right]\mathrm{d}V, \tag{5.1}$$

where $\mathbf{D}$ is the rate of deformation tensor, the symmetric part of the velocity gradient $\mathbf{L} = \mathbf{v}\otimes\nabla$. If $\mathbf{q}$ is the rate of heat flow by conduction across the surface element $\mathbf{n}\,\mathrm{d}S$, and w is the rate of heat input per unit current mass due to distributed internal heat sources, the total heat input rate is

$$\mathscr{Q} = -\int_S \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}S + \int_V \rho w\,\mathrm{d}V = \int_V (-\nabla\cdot\mathbf{q} + \rho w)\,\mathrm{d}V. \tag{5.2}$$

The first law of thermodynamics (conservation of energy) for the continuum with a growing mass can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + u\right)\mathrm{d}V = \mathscr{P} + \mathscr{Q} + \int_V r_\mathrm{g}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + u\right)\mathrm{d}V + \int_V \rho\mathscr{R}_\mathrm{g}r_\mathrm{g}\,\mathrm{d}V. \tag{5.3}$$

The third term on the right-hand side is the rate of kinetic and internal (u) energy associated with the current mass growth. The last term represents an average rate of chemical energy associated with the mass growth, in the case when the growth involves a deposition of different species. We introduced the affinity $\mathscr{R}_\mathrm{g}$, conjugate to the flux r_g, such that $\mathscr{R}_\mathrm{g}r_\mathrm{g}$ represents the rate of chemical energy per unit current mass (see, for example, Fung, 1990). Since by the Reynolds transport theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + u\right)\mathrm{d}V = \int_V \rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + u\right)\mathrm{d}V + \int_V r_\mathrm{g}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + u\right)\mathrm{d}V, \tag{5.4}$$

the substitution of Eqs. (5.1), (5.2) and (5.4) into Eq. (5.3) yields the local form of energy equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{\rho}\boldsymbol{\sigma}:\mathbf{D} - \frac{1}{\rho}\nabla\cdot\mathbf{q} + w + \mathscr{R}_\mathrm{g}r_\mathrm{g}. \tag{5.5}$$

This can also be deduced directly by applying the energy balance to an infinitesimal parallelepiped with the sides along the coordinate directions $\mathbf{e}_i$. The net rate of work of the traction vectors over all sides of the element is

$$\frac{\partial}{\partial x_i}(\mathbf{t}_i\cdot\mathbf{v}) = \nabla\cdot(\boldsymbol{\sigma}\cdot\mathbf{v}), \tag{5.6}$$

multiplied by the volume $\mathrm{d}V$, and from

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + u\right)\rho\,\mathrm{d}V\right] = \left[\frac{\partial}{\partial x_i}(\mathbf{t}_i\cdot\mathbf{v}) + \rho\mathbf{b}\cdot\mathbf{v} - \nabla\cdot\mathbf{q} + \rho w + r_\mathrm{g}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + u\right) + \rho\mathscr{R}_\mathrm{g}r_\mathrm{g}\right]\mathrm{d}V \tag{5.7}$$

we deduce by differentiation the energy equation (5.5).

5.1. Material form of energy equation

If the initial configuration is used to cast the expressions, we have

$$\mathscr{P} = \int_{V^0}\left[\rho_\mathrm{g}^0\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) + \mathbf{P}\cdots\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t}\right]\mathrm{d}V^0, \tag{5.8}$$

$$\mathscr{Q} = \int_{V^0} \left(- \nabla^0 \cdot \mathbf{q}^0 + \rho^0 w^0 \right) \mathrm{d}V^0, \tag{5.9}$$

where ∇^0 is the gradient operator with respect to initial coordinates, and $\mathbf{q}^0 = J\mathbf{F}^{-1} \cdot \mathbf{q}$ is the nominal heat flux vector. The energy equation for the whole continuum is then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V^0} \left[\rho_{\mathrm{g}}^0 \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) + \rho^0 u^0 \right] \mathrm{d}V^0 = \mathscr{P} + \mathscr{Q} + \int_{V^0} r_{\mathrm{g}}^0 \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \frac{\rho^0}{\rho_{\mathrm{g}}^0} u^0 \right) \mathrm{d}V^0 + \int_{V^0} \rho^0 \mathscr{R}_{\mathrm{g}}^0 r_{\mathrm{g}}^0 \, \mathrm{d}V^0. \tag{5.10}$$

The internal energy and the internal heat rate per unit initial mass are denoted by u^0 and w^0, such that

$$\rho^0 u^0 = \rho_{\mathrm{g}}^0 u, \quad \rho^0 w^0 = \rho_{\mathrm{g}}^0 w. \tag{5.11}$$

Performing the differentiation on the left-hand side of Eq. (5.10) and substituting Eqs. (5.8) and (5.9) yields

$$\frac{\mathrm{d}u^0}{\mathrm{d}t} - \frac{r_{\mathrm{g}}^0}{\rho_{\mathrm{g}}^0} u^0 = \frac{1}{\rho^0} \mathbf{P} \cdots \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} - \frac{1}{\rho^0} \nabla^0 \cdot \mathbf{q}^0 + w^0 + \mathscr{R}_{\mathrm{g}}^0 r_{\mathrm{g}}^0, \tag{5.12}$$

which is a dual equation to energy equation (5.5). If there is no mass growth, $r_{\mathrm{g}}^0 = 0$ and Eq. (5.12) reduces to the classical expression for the material form of energy equation (e.g., Truesdell and Toupin, 1960).

The affinities $\mathscr{R}_{\mathrm{g}}$ and $\mathscr{R}_{\mathrm{g}}^0$ are related by

$$\mathscr{M} \frac{\mathrm{d}}{\mathrm{d}t} (\mathrm{d}m) = \rho \mathscr{R}_{\mathrm{g}} (r_{\mathrm{g}} \mathrm{d}V) = \rho^0 \mathscr{R}_{\mathrm{g}}^0 (r_{\mathrm{g}}^0 \mathrm{d}V^0), \tag{5.13}$$

where $\mathscr{M}$ is the affinity conjugate to the mass flux, so that

$$\rho \mathscr{R}_{\mathrm{g}} = \rho^0 \mathscr{R}_{\mathrm{g}}^0. \tag{5.14}$$

Also, there is a connection between the rates of internal energy

$$\rho_{\mathrm{g}}^0 \frac{\mathrm{d}u}{\mathrm{d}t} = \rho^0 \left(\frac{\mathrm{d}u^0}{\mathrm{d}t} - \frac{r_{\mathrm{g}}^0}{\rho_{\mathrm{g}}^0} u^0 \right) = \rho^0 \left(\frac{\mathrm{d}u}{\mathrm{d}t} \right)^0. \tag{5.15}$$

This follows by differentiation from the first of Eq. (5.11), or from the general results (2.25) and (2.29).

6. Entropy equation

Let the rate of dissipation due to the rate of mass growth be

$$\mathscr{T}_{\mathrm{g}} \frac{\mathrm{d}}{\mathrm{d}t} (\mathrm{d}m) = \rho \Gamma_{\mathrm{g}} r_{\mathrm{g}} \mathrm{d}V, \quad \mathscr{T}_{\mathrm{g}} = \rho \Gamma_{\mathrm{g}}. \tag{6.1}$$

Suppose that ξ_v $(v = 1, 2, \ldots, n)$ are the internal variables that describe in some average sense the microstructural changes that occurred at the considered material particle during the deformation process. These, for example, can be used to describe the local structural remodeling caused by deformation and growth. Conceptually similar variables are used in the thermodynamic analysis of inelastic deformation processes of metals and other materials (Rice, 1971). The rate of dissipation due to structural changes, per unit current mass, can be expressed as

$$f_v \frac{\mathrm{d}\xi_v}{\mathrm{d}t} \quad (\text{sum on } v), \tag{6.2}$$

where f_v are the thermodynamic forces (affinities) conjugate to the fluxes $\mathrm{d}\xi_v/\mathrm{d}t$, similarly as Γ_g is conjugate to r_g. The total rate of dissipation, which is the product of the absolute temperature θ and the entropy production rate γ (per unit current mass) is then

$$\theta\gamma = \Gamma_\mathrm{g} r_\mathrm{g} + f_v \frac{\mathrm{d}\xi_v}{\mathrm{d}t}. \tag{6.3}$$

The second law of thermodynamics requires that $\gamma > 0$.

The integral form of the entropy equation for the continuum with a growing mass is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\eta\,\mathrm{d}V = -\int_S \frac{1}{\theta}\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}S + \int_V \frac{w}{\theta}\rho\,\mathrm{d}V + \int_V r_\mathrm{g}\eta\,\mathrm{d}V + \int_V \rho\gamma\,\mathrm{d}V, \tag{6.4}$$

where η stands for the entropy per unit current mass. Upon applying the Reynolds transport theorem to the left-hand side of Eq. (6.4), there follows

$$\int_V \rho\frac{\mathrm{d}\eta}{\mathrm{d}t}\,\mathrm{d}V = -\int_S \frac{1}{\theta}\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}S + \int_V \frac{w}{\theta}\rho\,\mathrm{d}V + \int_V \rho\gamma\,\mathrm{d}V. \tag{6.5}$$

This leads to a local form of the entropy equation

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\frac{1}{\rho}\nabla\cdot\left(\frac{1}{\theta}\mathbf{q}\right) + \frac{1}{\theta}\left(w + \Gamma_\mathrm{g} r_\mathrm{g} + f_v\frac{\mathrm{d}\xi_v}{\mathrm{d}t}\right). \tag{6.6}$$

If the temperature gradients are negligible,

$$\theta\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\frac{1}{\rho}\nabla\cdot\mathbf{q} + w + \Gamma_\mathrm{g} r_\mathrm{g} + f_v\frac{\mathrm{d}\xi_v}{\mathrm{d}t}. \tag{6.7}$$

6.1. Material form of entropy equation

The rate of dissipation per unit initial mass is

$$\theta\gamma^0 = \Gamma^0_\mathrm{g} r^0_\mathrm{g} + f^0_v\frac{\mathrm{d}\xi^0_v}{\mathrm{d}t}. \tag{6.8}$$

The relationships with the quantities per unit current mass are

$$\rho^0\Gamma^0_\mathrm{g} = \rho\Gamma_\mathrm{g}, \quad \rho^0 f^0_v = \rho f_v, \quad \frac{\mathrm{d}\xi^0_v}{\mathrm{d}t} = J\frac{\mathrm{d}\xi_v}{\mathrm{d}t}. \tag{6.9}$$

If η^0 is the entropy per unit initial mass, the integral form of the entropy equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V^0}\rho^0\eta^0\,\mathrm{d}V^0 = -\int_{S^0}\frac{1}{\theta}\mathbf{q}^0\cdot\mathbf{n}^0\,\mathrm{d}S^0 + \int_{V^0}\frac{w^0}{\theta}\rho^0\,\mathrm{d}V^0 + \int_{V^0} r^0_\mathrm{g}\frac{\rho^0}{\rho^0_\mathrm{g}}\eta^0\,\mathrm{d}V^0 + \int_{V^0}\rho^0\gamma^0\,\mathrm{d}V^0. \tag{6.10}$$

The corresponding local equation is

$$\frac{\mathrm{d}\eta^0}{\mathrm{d}t} - \frac{r^0_\mathrm{g}}{\rho^0_\mathrm{g}}\eta^0 = -\frac{1}{\rho^0}\nabla^0\cdot\left(\frac{1}{\theta}\mathbf{q}^0\right) + \frac{1}{\theta}\left(w^0 + \Gamma^0_\mathrm{g} r^0_\mathrm{g} + f^0_v\frac{\mathrm{d}\xi^0_v}{\mathrm{d}t}\right). \tag{6.11}$$

If the temperature gradients are negligible, this reduces to

$$\theta\left(\frac{\mathrm{d}\eta^0}{\mathrm{d}t} - \frac{r^0_\mathrm{g}}{\rho^0_\mathrm{g}}\eta^0\right) = -\frac{1}{\rho^0}\nabla^0\cdot\mathbf{q}^0 + w^0 + \Gamma^0_\mathrm{g} r^0_\mathrm{g} + f^0_v\frac{\mathrm{d}\xi^0_v}{\mathrm{d}t}. \tag{6.12}$$

Observe that from Eq. (2.29)

$$\frac{\mathrm{d}\eta^0}{\mathrm{d}t} - \frac{r_{\mathrm{g}}^0}{\rho_{\mathrm{g}}^0}\eta^0 = \left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)^0 = \frac{\rho_{\mathrm{g}}^0}{\rho^0}\frac{\mathrm{d}\eta}{\mathrm{d}t}. \tag{6.13}$$

6.2. Combined energy and entropy equations

When Eq. (6.7) is combined with Eq. (5.5) there follows

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{\rho}\boldsymbol{\sigma} : \mathbf{D} + \theta\frac{\mathrm{d}\eta}{\mathrm{d}t} + \left(\mathscr{R}_{\mathrm{g}} - \Gamma_{\mathrm{g}}\right)r_{\mathrm{g}} - f_v\frac{\mathrm{d}\xi_v}{\mathrm{d}t}. \tag{6.14}$$

Dually, when Eq. (6.13) is combined with Eq. (5.12), we obtain an expression for the rate of internal energy per unit initial mass

$$\frac{\mathrm{d}u^0}{\mathrm{d}t} - \frac{r_{\mathrm{g}}^0}{\rho_{\mathrm{g}}^0}u^0 = \frac{1}{\rho^0}\mathbf{P}\cdots\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} + \theta\left(\frac{\mathrm{d}\eta^0}{\mathrm{d}t} - \frac{r_{\mathrm{g}}^0}{\rho_{\mathrm{g}}^0}\eta^0\right) + \left(\mathscr{R}_{\mathrm{g}}^0 - \Gamma_{\mathrm{g}}^0\right)r_{\mathrm{g}}^0 - f_v^0\frac{\mathrm{d}\xi_v^0}{\mathrm{d}t}. \tag{6.15}$$

It is noted that

$$\Pi = \frac{1}{\rho}\boldsymbol{\sigma} : \mathbf{D} = \frac{1}{\rho_{\mathrm{g}}^0}\boldsymbol{\tau} : \mathbf{D}, \quad \boldsymbol{\tau} = J\boldsymbol{\sigma}, \tag{6.16}$$

is the stress power per unit current mass, while

$$\Pi^0 = \frac{1}{\rho^0}\mathbf{P}\cdots\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} \tag{6.17}$$

is the stress power per unit initial mass. These quantities are, of course, different and related by $\rho^0\Pi^0 = \rho_{\mathrm{g}}^0\Pi$. It is also recalled that the stress $\mathbf{T}$ conjugate to the material strain $\mathbf{E}$ is defined by (Hill, 1978)

$$\mathbf{P}\cdots\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} = \boldsymbol{\tau} : \mathbf{D} = \mathbf{T} : \frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t}. \tag{6.18}$$

7. General constitutive framework

Suppose that the internal energy per unit current mass is given by a function

$$u = u\left(\mathbf{E}, \eta, \rho_{\mathrm{g}}^0, \xi_v\right). \tag{7.1}$$

Its time rate is

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial \mathbf{E}} : \frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t} + \frac{\partial u}{\partial \eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} + \frac{\partial u}{\partial \rho_{\mathrm{g}}^0}r_{\mathrm{g}}^0 + \frac{\partial u}{\partial \xi_v}\frac{\mathrm{d}\xi_v}{\mathrm{d}t}. \tag{7.2}$$

When this is compared to (6.14) there follows

$$\mathbf{T} = \rho_{\mathrm{g}}^0\frac{\partial u}{\partial \mathbf{E}}, \tag{7.3}$$

$$\theta = \frac{\partial u}{\partial \eta}, \tag{7.4}$$

$$\mathscr{R}_{\mathrm{g}} - \Gamma_{\mathrm{g}} = J\,\frac{\partial u}{\partial \rho_{\mathrm{g}}^{0}}, \tag{7.5}$$

$$f_{v} = -\frac{\partial u}{\partial \xi_{v}}. \tag{7.6}$$

On the other hand, by introducing the Helmholtz free energy

$$\psi\left(\mathbf{E}, \theta, \rho_{\mathrm{g}}^{0}, \xi_{v}\right) = u\left(\mathbf{E}, \eta, \rho_{\mathrm{g}}^{0}, \xi_{v}\right) - \theta\eta, \tag{7.7}$$

we have by differentiation and incorporation of Eq. (6.14)

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{1}{\rho_{\mathrm{g}}^{0}}\mathbf{T} : \frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t} - \eta\,\frac{\mathrm{d}\theta}{\mathrm{d}t} + (\mathscr{R}_{\mathrm{g}} - \Gamma_{\mathrm{g}})r_{\mathrm{g}} - f_{v}\,\frac{\mathrm{d}\xi_{v}}{\mathrm{d}t}. \tag{7.8}$$

Thus, ψ is a thermodynamic potential such that

$$\mathbf{T} = \rho_{\mathrm{g}}^{0}\,\frac{\partial \psi}{\partial \mathbf{E}}, \tag{7.9}$$

$$\eta = -\frac{\partial \psi}{\partial \theta}, \tag{7.10}$$

$$\mathscr{R}_{\mathrm{g}} - \Gamma_{\mathrm{g}} = J\,\frac{\partial \psi}{\partial \rho_{\mathrm{g}}^{0}}, \tag{7.11}$$

$$f_{v} = -\frac{\partial \psi}{\partial \xi_{v}}. \tag{7.12}$$

The Maxwell-type relationships hold

$$\frac{\partial \mathbf{T}}{\partial \theta} = -\rho_{\mathrm{g}}^{0}\,\frac{\partial \eta}{\partial \mathbf{E}}, \quad \frac{\partial \mathbf{T}}{\partial \eta} = \rho_{\mathrm{g}}^{0}\,\frac{\partial \theta}{\partial \mathbf{E}}. \tag{7.13}$$

and

$$\frac{\partial \mathbf{T}}{\partial \xi_{v}} = -\rho_{\mathrm{g}}^{0}\,\frac{\partial f_{v}}{\partial \mathbf{E}}, \quad \frac{\partial}{\partial \mathbf{E}}[\rho(\mathscr{R}_{\mathrm{g}} - \Gamma_{\mathrm{g}})] = \frac{\partial \mathbf{T}}{\partial \rho_{\mathrm{g}}^{0}} - \frac{\mathbf{T}}{\rho_{\mathrm{g}}^{0}}. \tag{7.14}$$

7.1. Thermodynamic potentials per unit initial mass

In an alternative formulation we can introduce the internal energy per unit initial mass as

$$u^{0} = u^{0}\left(\mathbf{F}, \eta^{0}, \rho_{\mathrm{g}}^{0}, \xi_{v}^{0}\right). \tag{7.15}$$

The function u^0 is an objective function of $\mathbf{F}$, e.g., dependent on the right Cauchy–Green deformation tensor $\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}$. Its time rate is

$$\frac{\mathrm{d}u^{0}}{\mathrm{d}t} = \frac{\partial u^{0}}{\partial \mathbf{F}} \cdots \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} + \frac{\partial u^{0}}{\partial \eta^{0}}\,\frac{\mathrm{d}\eta^{0}}{\mathrm{d}t} + \frac{\partial u^{0}}{\partial \rho_{\mathrm{g}}^{0}}\,r_{\mathrm{g}}^{0} + \frac{\partial u^{0}}{\partial \xi_{v}^{0}}\,\frac{\mathrm{d}\xi_{v}^{0}}{\mathrm{d}t}. \tag{7.16}$$

The comparison with Eq. (6.15) establishes the constitutive structures

$$\mathbf{P} = \rho^0 \frac{\partial u^0}{\partial \mathbf{F}}, \tag{7.17}$$

$$\theta = \frac{\partial u^0}{\partial \eta^0}, \tag{7.18}$$

$$\mathscr{R}_{\mathrm{g}}^0 - \Gamma_{\mathrm{g}}^0 = \frac{\partial u^0}{\partial \rho_{\mathrm{g}}^0} - \frac{1}{\rho_{\mathrm{g}}^0}\left(u^0 - \theta\eta^0\right), \tag{7.19}$$

$$f_v^0 = -\frac{\partial u^0}{\partial \xi_v^0}. \tag{7.20}$$

If the Helmholtz free energy per unit initial mass is selected as a thermodynamic potential,

$$\psi^0(\mathbf{F}, \theta, \rho_{\mathrm{g}}^0, \xi_v^0) = u^0(\mathbf{F}, \eta^0, \rho_{\mathrm{g}}^0, \xi_v^0) - \theta\eta^0, \tag{7.21}$$

we obtain by differentiation and incorporation of Eq. (6.15)

$$\frac{\mathrm{d}\psi^0}{\mathrm{d}t} - \frac{r_{\mathrm{g}}^0}{\rho_{\mathrm{g}}^0}\psi^0 = \frac{1}{\rho^0}\mathbf{P} \cdots \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} - \eta^0 \frac{\mathrm{d}\theta}{\mathrm{d}t} + \left(\mathscr{R}_{\mathrm{g}}^0 - \Gamma_{\mathrm{g}}^0\right) r_{\mathrm{g}}^0 - f_v^0 \frac{\mathrm{d}\xi_v^0}{\mathrm{d}t}. \tag{7.22}$$

Since by partial differentiation

$$\frac{\mathrm{d}\psi^0}{\mathrm{d}t} = \frac{\partial \psi^0}{\partial \mathbf{F}} \cdots \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} + \frac{\partial \psi^0}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\partial \psi^0}{\partial \rho_{\mathrm{g}}^0} r_{\mathrm{g}}^0 + \frac{\partial \psi^0}{\partial \xi_v^0}\frac{\mathrm{d}\xi_v^0}{\mathrm{d}t}, \tag{7.23}$$

there follows

$$\mathbf{P} = \rho^0 \frac{\partial \psi^0}{\partial \mathbf{F}}, \tag{7.24}$$

$$\eta^0 = -\frac{\partial \psi^0}{\partial \theta}, \tag{7.25}$$

$$\mathscr{R}_{\mathrm{g}}^0 - \Gamma_{\mathrm{g}}^0 = \frac{\partial \psi^0}{\partial \rho_{\mathrm{g}}^0} - \frac{1}{\rho_{\mathrm{g}}^0}\psi^0, \tag{7.26}$$

$$f_v = -\frac{\partial \psi^0}{\partial \xi_v^0}. \tag{7.27}$$

7.2. Equivalence of the constitutive structures

The equivalence of the constitutive structures such as (7.3) and (7.17), or (7.4) and (7.18) is easily verified. The equivalence of (7.5) and (7.19) is less transparent and merits an explicit demonstration. To that goal, write the internal energy per unit initial mass as

$$u^0 = \frac{\rho_{\mathrm{g}}^0}{\rho^0} u\left(\mathbf{E}, \eta, \rho_{\mathrm{g}}^0, \xi_v\right) = \frac{\rho_{\mathrm{g}}^0}{\rho^0} u\left(\mathbf{E}, \frac{\rho^0}{\rho_{\mathrm{g}}^0}\eta^0, \rho_{\mathrm{g}}^0, \xi_v\right). \tag{7.28}$$

By taking the gradient with respect to ρ_{g}^0 it follows that

$$\frac{\partial u^0}{\partial \rho_{\mathrm{g}}^0} = \frac{1}{\rho^0} u - \frac{1}{\rho_{\mathrm{g}}^0} \eta^0 \frac{\partial u}{\partial \eta} + \frac{\rho_{\mathrm{g}}^0}{\rho^0} \frac{\partial u}{\partial \rho_{\mathrm{g}}^0}. \tag{7.29}$$

This can be rewritten as

$$\frac{\partial u^0}{\partial \rho_{\mathrm{g}}^0} = \frac{1}{\rho_{\mathrm{g}}^0} \left(u^0 - \theta \eta^0 \right) + \frac{\rho_{\mathrm{g}}^0}{\rho^0} \frac{1}{J} (\mathscr{R}_{\mathrm{g}} - \Gamma_{\mathrm{g}}), \tag{7.30}$$

in view of Eqs. (5.11), (7.4) and (7.5). Recalling that

$$\rho^0 \left(\mathscr{R}_{\mathrm{g}}^0 - \Gamma_{\mathrm{g}}^0 \right) = \rho (\mathscr{R}_{\mathrm{g}} - \Gamma_{\mathrm{g}}). \tag{7.31}$$

Eq. (7.30) becomes

$$\frac{\partial u^0}{\partial \rho_{\mathrm{g}}^0} = \frac{1}{\rho_{\mathrm{g}}^0} \left(u^0 - \theta \eta^0 \right) + \mathscr{R}_{\mathrm{g}}^0 - \Gamma_{\mathrm{g}}^0, \tag{7.32}$$

which is equivalent to Eq. (7.19). Similar derivation proceeds to establish the equivalence of Eqs. (7.11) and (7.26).

8. Multiplicative decomposition of deformation gradient

Let $\mathscr{B}_0$ be the initial configuration of the material sample, which is assumed to be stress free. If the original material sample supported a residual distribution of internal stress, we can imagine that the sample is dissected into small pieces to relieve the residual stress. In this case $\mathscr{B}_0$ is an incompatible configuration. For the constitutive analysis, however, this does not pose a problem, because it is sufficient to analyze any one of the stress-free pieces. Let $\mathbf{F}$ be a local deformation gradient that relates an infinitesimal material element $\mathrm{d}\mathbf{X}$ from $\mathscr{B}_0$ to $\mathrm{d}\mathbf{x}$ in the deformed configuration $\mathscr{B}$ at time t,

$$\mathrm{d}\mathbf{x} = \mathbf{F} \cdot \mathrm{d}\mathbf{X}. \tag{8.1}$$

The deformation gradient $\mathbf{F}$ is produced by the mass growth and deformation due to externally applied and growth induced stress. Introduce an intermediate configuration $\mathscr{B}_{\mathrm{g}}$ by instantaneous elastic distressing of the current configuration $\mathscr{B}$ to zero stress (Fig. 1). Define a local elastic deformation gradient $\mathbf{F}_{\mathrm{e}}$ that maps an infinitesimal element $\mathrm{d}\mathbf{x}_{\mathrm{g}}$ from $\mathscr{B}_{\mathrm{g}}$ to $\mathrm{d}\mathbf{x}$ in $\mathscr{B}$, such that

$$\mathrm{d}\mathbf{x} = \mathbf{F}_{\mathrm{e}} \cdot \mathrm{d}\mathbf{x}_{\mathrm{g}}. \tag{8.2}$$

Similarly, define a local growth deformation gradient $\mathbf{F}_{\mathrm{g}}$ that maps an infinitesimal element $\mathrm{d}\mathbf{X}$ from $\mathscr{B}_0$ to $\mathrm{d}\mathbf{x}_{\mathrm{g}}$ in $\mathscr{B}_{\mathrm{g}}$, such that

$$\mathrm{d}\mathbf{x}_{\mathrm{g}} = \mathbf{F}_{\mathrm{g}} \cdot \mathrm{d}\mathbf{X}. \tag{8.3}$$

Substituting Eq. (8.3) into Eq. (8.2) and comparing with Eq. (8.1) establishes the multiplicative decomposition of the deformation gradient into its elastic and growth parts

$$\mathbf{F} = \mathbf{F}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{g}}. \tag{8.4}$$

This decomposition is formally analogous to the well-known decomposition of elastoplastic deformation gradient into its elastic and plastic parts (see Lee, 1969; Lubarda and Lee, 1981), and was first introduced in biomechanics by Rodrigez et al. (1994). For the modification of the decomposition using the residually stressed reference configurations (see Hoger, 1997; Johnson and Hoger, 1998; Hoger et al., submitted).

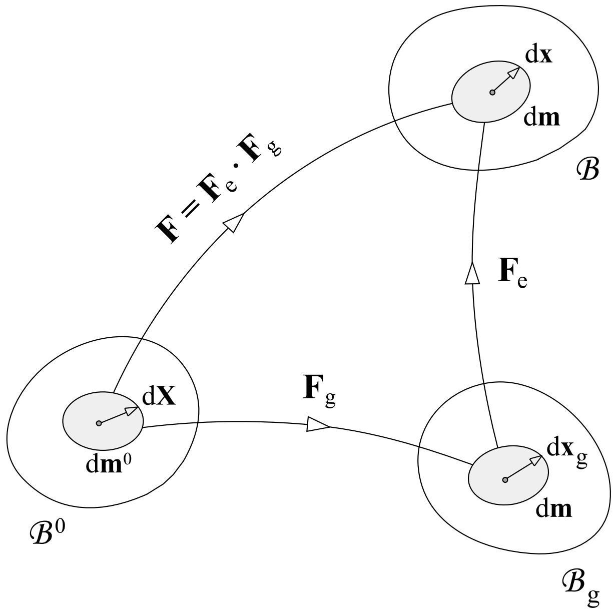

Fig. 1. Schematic representation of the multiplicative decomposition of deformation gradient into its elastic and growth parts. The intermediate configuration $\mathscr{B}_g$ is obtained from the current configuration $\mathscr{B}$ by instantaneous elastic destressing to zero stress. The mass of an infinitesimal volume element in $\mathscr{B}^0$ is $\mathrm{d}m^0$. The mass of the corresponding elements in $\mathscr{B}_g$ and $\mathscr{B}$ is $\mathrm{d}m$.

8.1. Non-uniqueness of decomposition

The deformation gradients $\mathbf{F}_e$ and $\mathbf{F}_g$ are not uniquely defined because arbitrary local material rotations can be superposed to unstressed intermediate configuration $\mathscr{B}_g$ preserving it unstressed. Thus, we can write

$$\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_g = \widehat{\mathbf{F}}_e \cdot \widehat{\mathbf{F}}_g, \tag{8.5}$$

where

$$\widehat{\mathbf{F}}_e = \mathbf{F}_e \cdot \widehat{\mathbf{Q}}^T, \quad \widehat{\mathbf{F}}_g = \widehat{\mathbf{Q}} \cdot \mathbf{F}_g. \tag{8.6}$$

The rotation tensor is represented by a proper orthogonal tensor $\widehat{\mathbf{Q}}$ ($\det \widehat{\mathbf{Q}} = 1$). Using the polar decompositions

$$\mathbf{F}_e = \mathbf{V}_e \cdot \mathbf{R}_e, \quad \mathbf{F}_g = \mathbf{R}_g \cdot \mathbf{U}_g, \tag{8.7}$$

we also have

$$\widehat{\mathbf{V}}_e = \mathbf{V}_e, \quad \widehat{\mathbf{R}}_e = \mathbf{R}_e \cdot \widehat{\mathbf{Q}}^T, \tag{8.8}$$

and

$$\widehat{\mathbf{R}}_g = \widehat{\mathbf{Q}} \cdot \mathbf{R}_g, \quad \widehat{\mathbf{U}}_g = \mathbf{U}_g. \tag{8.9}$$

Note that there is a unique rotation tensor $\mathbf{R}_{eg}$ appearing in the (unique) decomposition

$$\mathbf{F} = \mathbf{V}_e \cdot \mathbf{R}_{eg} \cdot \mathbf{U}_g, \quad \mathbf{R}_{eg} = \mathbf{R}_e \cdot \mathbf{R}_g, \tag{8.10}$$

since $\widehat{\mathbf{R}}_{eg} = \mathbf{R}_{eg}$.

The multiplicative decomposition (8.4) can be made unique by additional specifications dictated by a particular material model. For example, if the material is initially isotropic and the mass growth and stress-induced deformation do not change the state of isotropy, we can take $\mathbf{R}_e = \mathbf{I}$ (identity tensor) and $\mathbf{F}_e = \mathbf{V}_e$. On the other hand, if there is an orthogonal triad of preferred directions in the material (director vectors) which remains unaltered by the mass growth, we can uniquely define the intermediate configuration as one with the same orientation of director vectors as in the initial configuration. This latter choice is particularly appealing for biomechanical modeling and will be conveniently utilized in Sections 13–15.

8.2. Strain and strain-rate measures

A Lagrangian type strain measures associated with the deformation gradients $\mathbf{F}_e$ and $\mathbf{F}_g$ are

$$\mathbf{E}_e = \frac{1}{2}\left(\mathbf{F}_e^T \cdot \mathbf{F}_e - \mathbf{I}\right), \quad \mathbf{E}_g = \frac{1}{2}\left(\mathbf{F}_g^T \cdot \mathbf{F}_g - \mathbf{I}\right). \tag{8.11}$$

The total strain can be expressed in terms of these measures as

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}\right) = \mathbf{E}_g + \mathbf{F}_g^T \cdot \mathbf{E}_e \cdot \mathbf{F}_g. \tag{8.12}$$

Since $\mathbf{E}_e$ and $\mathbf{E}_g$ are defined with respect to different reference configurations, clearly $\mathbf{E} \neq \mathbf{E}_e + \mathbf{E}_g$. On the other hand, the velocity gradient becomes

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1} + \mathbf{F}_e \cdot \left(\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}\right) \cdot \mathbf{F}_e^{-1}. \tag{8.13}$$

The symmetric and antisymmetric parts of the second term on the far right-hand side will be conveniently denoted by

$$\mathbf{d}_g = \left[\mathbf{F}_e \cdot \left(\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}\right) \cdot \mathbf{F}_e^{-1}\right]_s, \quad \boldsymbol{\omega}_g = \left[\mathbf{F}_e \cdot \left(\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}\right) \cdot \mathbf{F}_e^{-1}\right]_a. \tag{8.14}$$

The rates of elastic and growth strains can now be expressed in terms of the rate of total strain $\dot{\mathbf{E}}$ and the velocity gradient of intermediate configuration $\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}$ as

$$\dot{\mathbf{E}}_e = \mathbf{F}_g^{-T} \cdot \dot{\mathbf{E}} \cdot \mathbf{F}_g^{-1} - \mathbf{F}_e^T \cdot \mathbf{d}_g \cdot \mathbf{F}_e, \tag{8.15}$$

$$\dot{\mathbf{E}}_g = \mathbf{F}_g^T \cdot \left(\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}\right)_s \cdot \mathbf{F}_g. \tag{8.16}$$

Naturally, these do not sum up to give the rate of total strain ($\dot{\mathbf{E}} \neq \dot{\mathbf{E}}_e + \dot{\mathbf{E}}_g$).

8.3. Objectivity considerations

Under a rigid-body rotation $\mathbf{Q}$ of the current configuration $\mathscr{B}$, the deformation gradient and its elastic and growth parts change to

$$\mathbf{F}^* = \mathbf{Q} \cdot \mathbf{F}, \quad \mathbf{F}_e^* = \mathbf{Q} \cdot \mathbf{F}_e \cdot \widehat{\mathbf{Q}}^T, \quad \mathbf{F}_g^* = \widehat{\mathbf{Q}} \cdot \mathbf{F}_g. \tag{8.17}$$

The rotation of the intermediate configuration, associated with the rotation $\mathbf{Q}$ of the current configuration, is denoted by $\widehat{\mathbf{Q}}$. The two rotations are identically equal ($\widehat{\mathbf{Q}} = \mathbf{Q}$) if the intermediate configuration is obtained by destressing without rotation ($\mathbf{F}_e = \mathbf{V}_e$), but $\widehat{\mathbf{Q}} = \mathbf{I}$ if the intermediate configuration is defined as one with the fixed orientation of the director vectors (in the context of elastoplastic deformations this was discussed by Lubarda (1991), and Lubarda and Shih (1994)). In either case, the following transformation rules apply

$$\mathbf{V}_e^* = \mathbf{Q} \cdot \mathbf{V}_e \cdot \mathbf{Q}^T, \quad \mathbf{U}_g^* = \mathbf{U}_g, \tag{8.18}$$

$$\mathbf{R}_e^* = \mathbf{Q} \cdot \mathbf{R}_e \cdot \widehat{\mathbf{Q}}^T, \quad \mathbf{R}_g^* = \widehat{\mathbf{Q}} \cdot \mathbf{R}_g, \quad \mathbf{R}_{eg}^* = \mathbf{Q} \cdot \mathbf{R}_{eg}, \tag{8.19}$$

$$\mathbf{E}_e^* = \widehat{\mathbf{Q}} \cdot \mathbf{E}_e \cdot \widehat{\mathbf{Q}}^T, \quad \mathbf{E}_g^* = \mathbf{E}_g. \tag{8.20}$$

The rates of strain transform according to

$$\dot{\mathbf{E}}_e^* = \widehat{\mathbf{Q}} \cdot \dot{\mathbf{E}}_e \cdot \widehat{\mathbf{Q}}^T + \left(\dot{\widehat{\mathbf{Q}}} \cdot \widehat{\mathbf{Q}}^{-1}\right) \cdot \mathbf{E}_e^* - \mathbf{E}_e^* \cdot \left(\dot{\widehat{\mathbf{Q}}} \cdot \widehat{\mathbf{Q}}^{-1}\right), \tag{8.21}$$

$$\dot{\mathbf{E}}_g^* = \dot{\mathbf{E}}_g, \quad \dot{\mathbf{E}}^* = \dot{\mathbf{E}}. \tag{8.22}$$

9. Density expressions

If the mass of an infinitesimal volume element in the initial configuration is $\mathrm{d}m^0 = \rho^0 \,\mathrm{d}V^0$, the mass of the corresponding element in the configurations $\mathscr{B}_g$ and $\mathscr{B}$ is

$$\mathrm{d}m = \rho_g \,\mathrm{d}V_g = \rho \,\mathrm{d}V. \tag{9.1}$$

Since

$$\mathrm{d}m = \mathrm{d}m^0 + \int_0^t r_g^0 \,\mathrm{d}\tau \,\mathrm{d}V^0, \tag{9.2}$$

and

$$\mathrm{d}V_g = J_g \,\mathrm{d}V^0, \quad J_g = \det \mathbf{F}_g, \tag{9.3}$$

we have

$$\rho_g J_g = \rho^0 + \int_0^t r_g^0 \,\mathrm{d}\tau. \tag{9.4}$$

In addition, from Eq. (9.1) there follows

$$\rho_g J_g = \rho J, \quad \rho_g = \rho J_e, \tag{9.5}$$

because $\mathrm{d}V = J_e \,\mathrm{d}V_g$ and $J = J_e J_g$, where $J_e = \det \mathbf{F}_e$.

If $\rho_g = \rho^0$ throughout the mass growth, Eq. (9.4) reduces to

$$J_g = 1 + \frac{1}{\rho^0} \int_0^t r_g^0 \,\mathrm{d}\tau. \tag{9.6}$$

For elastically incompressible material, $J_e = 1$ and $\rho = \rho_g$.

From Eqs. (9.4) and (9.5) we further have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho J) = \frac{\mathrm{d}}{\mathrm{d}t}(\rho_g J_g) = r_g^0. \tag{9.7}$$

This yields the continuity equations for the densities ρ and ρ_g, i.e.,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \,\mathrm{tr}\left(\dot{\mathbf{F}} \cdot \mathbf{F}^{-1}\right) = r_g, \tag{9.8}$$

$$\frac{\mathrm{d}\rho_g}{\mathrm{d}t} + \rho_g \,\mathrm{tr}\left(\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}\right) = r_g J_e. \tag{9.9}$$

In view of the additive decomposition

$$\mathrm{tr}\left(\dot{\mathbf{F}}\cdot\mathbf{F}^{-1}\right)=\mathrm{tr}\left(\dot{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1}\right)+\mathrm{tr}\left(\dot{\mathbf{F}}_{\mathrm{g}}\cdot\mathbf{F}_{\mathrm{g}}^{-1}\right), \tag{9.10}$$

which results from Eq. (8.13) and the cyclic property of the trace of a matrix product, from Eqs. (9.8) and (9.9) it readily follows that

$$J_{\mathrm{e}}\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho_{\mathrm{g}}\,\mathrm{tr}\left(\dot{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1}\right)=\frac{\mathrm{d}\rho_{\mathrm{g}}}{\mathrm{d}t}. \tag{9.11}$$

If the growth takes place in a density preserving manner ($\rho_{\mathrm{g}}=\rho^0=\mathrm{const.}$), we have $\rho J=\rho^0 J_{\mathrm{g}}$, $\rho^0=\rho J_{\mathrm{e}}$, and

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\,\mathrm{tr}\left(\dot{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1}\right)=0, \tag{9.12}$$

$$\rho\,\mathrm{tr}\left(\dot{\mathbf{F}}_{\mathrm{g}}\cdot\mathbf{F}_{\mathrm{g}}^{-1}\right)=r_{\mathrm{g}}. \tag{9.13}$$

The last expression also follows directly from Eq. (2.1) by using $\mathrm{d}m=\rho^0\,\mathrm{d}V_{\mathrm{g}}$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}V_{\mathrm{g}})=\mathrm{tr}\left(\dot{\mathbf{F}}_{\mathrm{g}}\cdot\mathbf{F}_{\mathrm{g}}^{-1}\right)\mathrm{d}V_{\mathrm{g}}. \tag{9.14}$$

If material is also elastically incompressible, $\mathrm{tr}\left(\dot{\mathbf{F}}_{\mathrm{e}}\cdot\mathbf{F}_{\mathrm{e}}^{-1}\right)=0$ and $\mathrm{d}\rho/\mathrm{d}t=0$ (i.e., $\rho=\rho^0=\mathrm{const.}$).

In the case when the mass growth occurs by densification only, i.e. when $\mathrm{d}V_{\mathrm{g}}=\mathrm{d}V^0$ (volume preserving mass growth), we have $J_{\mathrm{g}}=1$ and

$$\mathrm{tr}\left(\dot{\mathbf{F}}_{\mathrm{g}}\cdot\mathbf{F}_{\mathrm{g}}^{-1}\right)=0,\quad \frac{\mathrm{d}\rho_{\mathrm{g}}}{\mathrm{d}t}=r_{\mathrm{g}}J. \tag{9.15}$$

If, in addition, the material is elastically incompressible, then $\mathrm{d}V=\mathrm{d}V_{\mathrm{g}}=\mathrm{d}V^0$, $J=1$, $\rho=\rho_{\mathrm{g}}$, and

$$\mathrm{tr}\left(\dot{\mathbf{F}}\cdot\mathbf{F}^{-1}\right)=0,\quad \frac{\mathrm{d}\rho}{\mathrm{d}t}=r_{\mathrm{g}}. \tag{9.16}$$

This, for example, could occur in an incompressible tissue which increases its mass by an increasing concentration of collagen molecules. In some cases, on the other hand, it may be unlikely that the material can increase its mass by densification only, while being elastically incompressible. More often, the mass growth by densification occurs in porous materials, which are commonly characterized by elastic compressibility.

10. Elastic stress response

Consider an isothermal deformation and growth process. Denote the set of structural tensors that describe the state of elastic anisotropy in both the initial and intermediate configurations by $\mathbf{S}^0$. For simplicity, we assume that the state of elastic anisotropy remains unaltered during the growth and deformation. The elastic strain energy per unit current mass is then given by an isotropic function of the elastic strain $\mathbf{E}_{\mathrm{e}}$ and the tensors $\mathbf{S}^0$, i.e.,

$$\psi\left(\mathbf{E}_{\mathrm{e}},\mathbf{S}^0,\rho_{\mathrm{g}}^0\right)=\psi\left[\mathbf{F}_{\mathrm{g}}^{-\mathrm{T}}\cdot(\mathbf{E}-\mathbf{E}_{\mathrm{g}})\cdot\mathbf{F}_{\mathrm{g}}^{-1},\mathbf{S}^0,\rho_{\mathrm{g}}^0\right]. \tag{10.1}$$

Introduce the stress tensor $\mathbf{T}_{\mathrm{e}}$ such that $\mathbf{T}_{\mathrm{e}}:\mathrm{d}\mathbf{E}_{\mathrm{e}}$ is the increment of elastic work per unit unstressed volume in the configuration $\mathscr{B}_{\mathrm{g}}$. Since $\rho_{\mathrm{g}}\psi$ is the elastic strain energy per unit unstressed volume, we can write

$$\mathbf{T}_{\mathrm{e}} : \mathrm{d}\mathbf{E}_{\mathrm{e}} = \frac{\partial(\rho_{\mathrm{g}}\psi)}{\partial \mathbf{E}_{\mathrm{e}}} : \mathrm{d}\mathbf{E}_{\mathrm{e}}, \quad \mathbf{T}_{\mathrm{e}} = \frac{\partial(\rho_{\mathrm{g}}\psi)}{\partial \mathbf{E}_{\mathrm{e}}}. \tag{10.2}$$

On the other hand, the strain energy per unit initial volume in the configuration $\mathscr{B}^0$ is $\rho_{\mathrm{g}}^0\psi$, and

$$\mathbf{T} : \mathrm{d}\mathbf{E} = \frac{\partial(\rho_{\mathrm{g}}^0\psi)}{\partial \mathbf{E}} : \mathrm{d}\mathbf{E}, \quad \mathbf{T} = \frac{\partial(\rho_{\mathrm{g}}^0\psi)}{\partial \mathbf{E}}, \tag{10.3}$$

in accord with Eq. (7.9). In view of Eq. (10.1), the partial differentiation gives

$$\mathbf{T} = \frac{\partial(\rho_{\mathrm{g}}^0\psi)}{\partial \mathbf{E}_{\mathrm{e}}} : \frac{\partial \mathbf{E}_{\mathrm{e}}}{\partial \mathbf{E}} = \mathbf{F}_{\mathrm{g}}^{-1} \cdot \frac{\partial(\rho_{\mathrm{g}}^0\psi)}{\partial \mathbf{E}_{\mathrm{e}}} \cdot \mathbf{F}_{\mathrm{g}}^{-\mathrm{T}}. \tag{10.4}$$

Since

$$\mathbf{T} = \mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}^{-\mathrm{T}}, \quad \boldsymbol{\tau} = J\boldsymbol{\sigma}, \tag{10.5}$$

combining Eqs. (10.4) and (10.5) and using the multiplicative decomposition $\mathbf{F} = \mathbf{F}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{g}}$, there follows

$$\mathbf{T}_{\mathrm{e}} = \mathbf{F}_{\mathrm{e}}^{-1} \cdot \boldsymbol{\tau}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-\mathrm{T}}, \quad \boldsymbol{\tau}_{\mathrm{e}} = J_{\mathrm{e}}\boldsymbol{\sigma}. \tag{10.6}$$

Thus, we have

$$\boldsymbol{\tau}_{\mathrm{e}} = \mathbf{F}_{\mathrm{e}} \cdot \frac{\partial(\rho_{\mathrm{g}}\psi)}{\partial \mathbf{E}_{\mathrm{e}}} \cdot \mathbf{F}_{\mathrm{e}}^{\mathrm{T}}, \tag{10.7}$$

and

$$\boldsymbol{\tau} = \mathbf{F} \cdot \frac{\partial(\rho_{\mathrm{g}}^0\psi)}{\partial \mathbf{E}} \cdot \mathbf{F}^{\mathrm{T}} = \mathbf{F}_{\mathrm{e}} \cdot \frac{\partial(\rho_{\mathrm{g}}^0\psi)}{\partial \mathbf{E}_{\mathrm{e}}} \cdot \mathbf{F}_{\mathrm{e}}^{\mathrm{T}}. \tag{10.8}$$

In terms of the right Cauchy–Green deformation tenors $\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}$ and $\mathbf{C}_{\mathrm{e}} = \mathbf{F}_{\mathrm{e}}^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{e}}$, this can be rewritten as

$$\boldsymbol{\tau} = 2\mathbf{F} \cdot \frac{\partial(\rho_{\mathrm{g}}^0\psi)}{\partial \mathbf{C}} \cdot \mathbf{F}^{\mathrm{T}} = 2\mathbf{F}_{\mathrm{e}} \cdot \frac{\partial(\rho_{\mathrm{g}}^0\psi)}{\partial \mathbf{C}_{\mathrm{e}}} \cdot \mathbf{F}_{\mathrm{e}}^{\mathrm{T}}. \tag{10.9}$$

11. Partition of the rate of deformation

The elastic part of the rate of deformation tensor will be defined by a kinetic relation

$$\mathbf{D}_{\mathrm{e}} = \mathscr{L}_{\mathrm{e}}^{-1} : \overset{\circ}{\boldsymbol{\tau}}, \tag{11.1}$$

where

$$\overset{\circ}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{W} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{W} \tag{11.2}$$

is the Jaumann derivative of the Kirchhoff stress relative to material spin $\mathbf{W}$, and $\mathscr{L}_{\mathrm{e}}$ is the corresponding fourth-order elastic moduli tensor. The remaining part of the rate of deformation will be referred to as the growth part of the rate of deformation tensor, such that

$$\mathbf{D} = \mathbf{D}_{\mathrm{e}} + \mathbf{D}_{\mathrm{g}}. \tag{11.3}$$

To derive an expression for $\mathbf{D}_g$, we differentiate the second of Eq. (10.8) and obtain

$$\dot{\boldsymbol{\tau}} = \left(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1}\right) \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \left(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1}\right)^{\mathrm{T}} + \mathbf{F}_e \cdot \left(\boldsymbol{\Lambda}_e : \dot{\mathbf{E}}_e\right) \cdot \mathbf{F}_e^{\mathrm{T}} + \frac{\partial \boldsymbol{\tau}}{\partial \rho_g^0} r_g^0, \tag{11.4}$$

where

$$\boldsymbol{\Lambda}_e = \frac{\partial^2(\rho_g^0 \psi)}{\partial \mathbf{E}_e \otimes \partial \mathbf{E}_e} = 4 \frac{\partial^2(\rho_g^0 \psi)}{\partial \mathbf{C}_e \otimes \partial \mathbf{C}_e}, \tag{11.5}$$

and

$$\frac{\partial \boldsymbol{\tau}}{\partial \rho_g^0} = \mathbf{F}_e \cdot \frac{\partial^2(\rho_g^0 \psi)}{\partial \mathbf{E}_e \, \partial \rho_g^0} \cdot \mathbf{F}_e^{\mathrm{T}} = 2\mathbf{F}_e \cdot \frac{\partial^2(\rho_g^0 \psi)}{\partial \mathbf{C}_e \, \partial \rho_g^0} \cdot \mathbf{F}_e^{\mathrm{T}}. \tag{11.6}$$

The structural tensors $\mathbf{S}^0$ remain constant during the differentiation. Since

$$\dot{\mathbf{E}}_e = \mathbf{F}_e^{\mathrm{T}} \cdot \left(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1}\right)_s \cdot \mathbf{F}_e, \tag{11.7}$$

the substitution into Eq. (11.4) gives

$$\dot{\boldsymbol{\tau}} - \left(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1}\right)_a \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \left(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1}\right)_a = \mathscr{L}_e : \left(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1}\right)_s + \frac{\partial \boldsymbol{\tau}}{\partial \rho_g^0} r_g^0. \tag{11.8}$$

The rectangular components of the elastic moduli tensor $\mathscr{L}_e$ are

$$\mathscr{L}^e_{ijkl} = F^e_{im} F^e_{jn} \Lambda^e_{mnpq} F^e_{kp} F^e_{lq} + \tfrac{1}{2}\left(\tau_{ik}\delta_{jl} + \tau_{jk}\delta_{il} + \tau_{il}\delta_{jk} + \tau_{jl}\delta_{ik}\right). \tag{11.9}$$

When the antisymmetric part of Eq. (8.13) is inserted into Eq. (11.9), there follows

$$\overset{\circ}{\boldsymbol{\tau}} = \mathscr{L}_e : \left(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1}\right)_s - \boldsymbol{\omega}_g \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\omega}_g + \frac{\partial \boldsymbol{\tau}}{\partial \rho_g^0} r_g^0. \tag{11.10}$$

By taking the symmetric part of Eq. (8.13) we have

$$\left(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1}\right)_s = \mathbf{D} - \mathbf{d}_g, \tag{11.11}$$

so that Eq. (11.10) can be rewritten as

$$\mathscr{L}_e^{-1} : \overset{\circ}{\boldsymbol{\tau}} = \mathbf{D} - \mathbf{d}_g - \mathscr{L}_e^{-1} : \left(\boldsymbol{\omega}_g \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}_g - \frac{\partial \boldsymbol{\tau}}{\partial \rho_g^0} r_g^0\right). \tag{11.12}$$

According to Eq. (11.1), the left-hand side is the elastic part of the rate of deformation tensor, so that the growth part is given by

$$\mathbf{D}_g = \mathbf{d}_g + \mathscr{L}_e^{-1} : \left(\boldsymbol{\omega}_g \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}_g - \frac{\partial \boldsymbol{\tau}}{\partial \rho_g^0} r_g^0\right). \tag{11.13}$$

For related analysis in the context of damage-elastoplasticity, see the papers by Lubarda (1994), and Lubarda and Krajcinovic (1995).

12. Isotropic materials

For isotropic materials that remain isotropic during the mass growth and deformation, the elastic strain energy is an isotropic function of elastic deformation tensor, i.e., the function of its principal invariants

$$\psi = \psi\left(\mathbf{C}_e, \rho_g^0\right) = \psi\left(I_C, II_C, III_C, \rho_g^0\right). \tag{12.1}$$

The principal invariants are (e.g., Ogden, 1984)

$$I_C = \operatorname{tr}\mathbf{C}_e, \quad II_C = \tfrac{1}{2}\left[\operatorname{tr}\left(\mathbf{C}_e^2\right) - (\operatorname{tr}\mathbf{C}_e)^2\right], \quad III_C = \det\mathbf{C}_e. \tag{12.2}$$

The Kirchhoff stress is

$$\boldsymbol{\tau} = 2\left(c_2\mathbf{I} + c_0\mathbf{B}_e + c_1\mathbf{B}_e^2\right), \tag{12.3}$$

where $\mathbf{B}_e = \mathbf{F}_e \cdot \mathbf{F}_e^T$ is the left Cauchy–Green deformation tensor, and

$$c_0 = \frac{\partial(\rho_g^0\psi)}{\partial I_C} - I_C\frac{\partial(\rho_g^0\psi)}{\partial II_C}, \quad c_1 = \frac{\partial(\rho_g^0\psi)}{\partial II_C}, \quad c_2 = III_C\frac{\partial(\rho_g^0\psi)}{\partial III_C}. \tag{12.4}$$

If the mass growth occurs isotropically, the growth part of the deformation gradient is

$$\mathbf{F}_g = \vartheta_g\mathbf{I}, \tag{12.5}$$

where ϑ_g is the isotropic stretch ratio due to volumetric mass growth. It readily follows that the velocity gradient in the intermediate configuration $\mathscr{B}_g$ is

$$\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1} = \frac{\dot{\vartheta}_g}{\vartheta_g}\mathbf{I}, \tag{12.6}$$

while in the configuration $\mathscr{B}$,

$$\mathbf{L} = \dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1} + \frac{\dot{\vartheta}_g}{\vartheta_g}\mathbf{I}. \tag{12.7}$$

Since $\boldsymbol{\omega}_g = \mathbf{0}$, the growth part of the rate of deformation tensor is

$$\mathbf{D}_g = \frac{\dot{\vartheta}_g}{\vartheta_g}\mathbf{I} - \mathscr{L}_e^{-1} : \left(\frac{\partial\boldsymbol{\tau}}{\partial\rho_g^0}\, r_g^0\right), \tag{12.8}$$

which follows from Eq. (11.13). The elastic part of the deformation gradient is

$$\mathbf{F}_e = \frac{1}{\vartheta_g}\mathbf{F}. \tag{12.9}$$

The rectangular components of the elastic moduli tensor $\boldsymbol{\Lambda}_e$ appearing in Eqs. (11.5) and (11.9), are

$$\Lambda_{ijkl}^e = 4[a_1\delta_{ij}\delta_{kl} + a_2\delta_{(ik}\delta_{jl)} + a_3\delta_{(ij}C_{kl)}^e + a_4C_{ij}^eC_{kl}^e + a_5\delta_{(ij}C_{kl)}^{e-1} + a_6C_{(ij}^eC_{kl)}^{e-1} + a_7C_{ij}^{e-1}C_{kl}^{e-1} + a_8C_{(ik}^{e-1}C_{jl)}^{e-1}]. \tag{12.10}$$

The coefficients a_i $(i = 1, 2, \ldots, 8)$ are defined in Appendix A. The symmetrization with respect to i and j, k and l, and ij and kl is used in Eq. (12.10), such that

$$\delta_{(ij}C_{kl)}^e = \tfrac{1}{2}\left(\delta_{ij}C_{kl}^e + C_{ij}^e\delta_{kl}\right), \tag{12.11}$$

and similarly for other terms.

In the case of elastically incompressible material, there is a geometric constraint $III_C = 1$, so that the Cauchy stress becomes

$$\boldsymbol{\sigma} = -p\mathbf{I} + \frac{2}{J}\left(c_0\mathbf{B}_e + c_1\mathbf{B}_e^2\right), \tag{12.12}$$

where p is an arbitrary pressure, indeterminate by the constitutive analysis. (In the unstressed configuration we take p to be equal to p_0 such that the overall stress is there equal to zero.) The rectangular components of the elastic moduli tensor $\mathbf{\Lambda}_e$ are given by Eq. (12.10) with the coefficients $a_5 = a_6 = a_7 = a_8 = 0$, i.e.,

$$\Lambda^e_{ijkl} = 4\left[a_1\delta_{ij}\delta_{kl} + a_2\delta_{(ik}\delta_{jl)} + a_3\delta_{(ij}C^e_{kl)} + a_4C^e_{ij}C^e_{kl}\right]. \tag{12.13}$$

12.1. Elastic strain energy representation

Various forms of the strain energy function were proposed in the literature for different biological materials. The articles by Holzapfel et al. (2000), and Sacks (2000) contain a number of pertinent references. Following Fung's (1973, 1995) proposal for a vascular soft tissue modeled as an incompressible elastic material, we consider the following structure of the elastic strain energy per unit initial volume

$$\rho^0_g\psi = \tfrac{1}{2}\alpha_0[\exp(Q) - Q - 1] + \tfrac{1}{2}q - \tfrac{1}{2}p(III_C - 1), \tag{12.14}$$

where Q and q are the polynomials in the invariants of $\mathbf{C}_e$ which include terms up to the fourth order in elastic stretch ratios, i.e.,

$$Q = \alpha_1(I_C - 3) + \alpha_2(II_C - 3) + \alpha_3(I_C - 3)^2, \tag{12.15}$$

$$q = \beta_1(I_C - 3) + \beta_2(II_C - 3) + \beta_3(I_C - 3)^2. \tag{12.16}$$

The incompressibility constraint in Eq. (12.14) is $III_C - 1 = 0$, and the pressure p plays the role of the Lagrangian multiplier. The αs and βs are the material parameters. In order that the intermediate configuration is unstressed, we require that $\beta_1 - 2\beta_2 = Jp_0$. If the material constants are such that $\beta_1 = 2\beta_2$, then $p_0 = 0$.

12.2. Evolution equation for stretch ratio

The constitutive formulation is completed by specifying an appropriate evolution equation for the stretch ratio ϑ_g. In a particular, but for the tissue mechanics important special case when the growth takes place in a density preserving manner ($\rho_g = \rho^0$), we have from Eqs. (9.13) and (12.6)

$$\mathrm{tr}\left(\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}\right) = 3\frac{\dot{\vartheta}_g}{\vartheta_g} = \frac{r_g}{\rho}. \tag{12.17}$$

Thus, recalling that $r_g/\rho = r^0_g/\rho^0_g$, the rate of mass growth $r^0_g = \mathrm{d}\rho^0_g/\mathrm{d}t$ can be expressed in terms of the rate of stretch $\dot{\vartheta}_g$ as

$$r^0_g = 3\rho^0_g\frac{\dot{\vartheta}_g}{\vartheta_g}. \tag{12.18}$$

Upon integration of Eq. (12.18) using the initial conditions $\vartheta^0_g = 1$ and $\rho^0_g = \rho^0$, we obtain

$$\rho^0_g = \rho^0\vartheta^3_g. \tag{12.19}$$

We propose as an evolution equation for the stretch ratio ϑ_g the following expression

$$\dot{\vartheta}_g = f_\vartheta(\vartheta_g, \mathrm{tr}\,\mathbf{T}_e). \tag{12.20}$$

The tensor $\mathbf{T}_e$ is the symmetric Piola–Kirchhoff stress with respect to intermediate configuration $\mathscr{B}_g$ where the stretch ratio ϑ_g is defined. For isotropic mass growth, only spherical part of this tensor is assumed to affect the change of the stretch ratio. In view of Eq. (10.6), this can be expressed in terms of the Cauchy stress $\boldsymbol{\sigma}$ and the elastic deformation as

$$\operatorname{tr}\mathbf{T}_e = J_e\mathbf{B}_e^{-1} : \boldsymbol{\sigma}. \tag{12.21}$$

The simplest evolution of the stretch ratio incorporates a linear dependence on stress, such that

$$\dot{\vartheta}_g = k_\vartheta(\vartheta_g)\operatorname{tr}\mathbf{T}_e. \tag{12.22}$$

This implies that the growth-equilibrium stress is equal to zero (i.e., $\dot{\vartheta}_g = 0$ when $\operatorname{tr}\mathbf{T}_e = 0$). The coefficient k_ϑ may be constant, or dependent on ϑ_g. For example, k_ϑ may take one value during the development of the tissue, and another value during the normal maturity. Yet another value may be characteristic for abnormal conditions, such as occur in thickening of blood vessels under hypertension. To prevent an unlimited growth at non-zero stress, we suggest the following expression for the function k_ϑ in Eq. (12.22)

$$k_\vartheta(\vartheta_g) = k_{\vartheta 0}^{+}\left(\frac{\vartheta_g^{+} - \vartheta_g}{\vartheta_g^{+} - 1}\right)^{m_\vartheta^{+}}, \quad \operatorname{tr}\mathbf{T}_e > 0, \tag{12.23}$$

where $\vartheta_g^{+} > 1$ is the limiting value of the stretch ratio that can be reached by mass growth, and $k_{\vartheta 0}^{+}$ and m_ϑ^{+} are the appropriate constants (material parameters). If the mass growth is homogeneous throughout the body, ϑ_g^{+} is constant, but for non-uniform mass growth caused by non-uniform biochemical properties ϑ_g^{+} may be different at different points (for example, inner and outer layers of the aorta may have different growth potentials, in addition to stress-modulated growth effects). We assume that the stress-modulated growth occurs under tension, while resorption takes place under compression. In the latter case

$$k_\vartheta(\vartheta_g) = k_{\vartheta 0}^{-}\left(\frac{\vartheta_g - \vartheta_g^{-}}{1 - \vartheta_g^{-}}\right)^{m_\vartheta^{-}}, \quad \operatorname{tr}\mathbf{T}_e < 0, \tag{12.24}$$

where $\vartheta_g^{-} < 1$ is the limiting value of the stretch ratio that can be reached by mass resorption. For generality, we assume that the resorption parameters $k_{\vartheta 0}^{-}$ and m_ϑ^{-} are different than those in growth. It should also be noted that other evolution equations were suggested motivated by the possibilities of growth and resorption. The most well known is the evolution equation for the mass growth in terms of a non-linear function of stress, which includes three growth-equilibrium states of stress (Fung, 1990)

12.3. Mass growth by densification

If the mass growth takes place by densification only, we have

$$\mathbf{F}_g = \mathbf{I}, \quad \mathbf{F} = \mathbf{F}_e, \tag{12.25}$$

and

$$\mathbf{D}_e = \mathscr{L}_e^{-1} : \overset{\circ}{\boldsymbol{\tau}}, \quad \mathbf{D}_g = -\mathscr{L}_e^{-1} : \left(\frac{\partial \boldsymbol{\tau}}{\partial \rho_g^0}\, r_g^0\right). \tag{12.26}$$

The growth part of the rate of deformation $\mathbf{D}_g$ in Eq. (12.26) can, for example, be related to the change of elastic properties caused by the densification. The Jaumann rate of the Kirchhoff stress can be expressed in terms of the rate of deformation tensor as

$$\overset{\circ}{\boldsymbol{\tau}} = \mathscr{L}_e : \mathbf{D} + \frac{\partial \boldsymbol{\tau}}{\partial \rho_g^0}\, r_g^0. \tag{12.27}$$

13. Transversely isotropic materials

A tissue containing a bundle of longitudinal fibers can be modeled as transversely isotropic. Let the unit vector $\mathbf{m}^0$ specify the fiber orientation in the initial configuration $\mathscr{B}^0$. The intermediate configuration $\mathscr{B}_g$ will be defined as one with the same fiber orientation relative to the fixed frame of reference (Fig. 2). Assuming that the fibers are imbedded in the material, this is ensured by defining $\mathbf{F}_g$ such that $\mathbf{m}^0$ is one of its eigendirections, i.e.,

$$\mathbf{F}_g \cdot \mathbf{m}^0 = \eta_g \mathbf{m}^0. \tag{13.1}$$

An infinitesimal fiber segment in the configuration $\mathscr{B}$ is parallel to the vector $\mathbf{m} = \mathbf{F}_e \cdot \mathbf{m}^0$, obtained from $\mathbf{m}^0$ by elastic stretching and rotation. The elastic strain energy per unit initial volume can be written as an isotropic function of both elastic deformation tensor $\mathbf{C}_e$ and the structural tensor $\mathbf{m}^0 \otimes \mathbf{m}^0$,

$$\rho_g^0 \psi = \rho_g^0 \psi\left(\mathbf{C}_e, \mathbf{m}^0 \otimes \mathbf{m}^0, \rho_g^0\right). \tag{13.2}$$

More specifically, the function ψ can be expressed in terms of individual and joint invariants of $\mathbf{C}_e$ and $\mathbf{m}^0$, such that

$$\psi = \psi\left(I_C, II_C, III_C, M_1, M_2, \rho_g^0\right). \tag{13.3}$$

The joint of mixed invariants are

$$M_1 = \mathbf{m}^0 \cdot \mathbf{C}_e \cdot \mathbf{m}^0, \quad M_2 = \mathbf{m}^0 \cdot \mathbf{C}_e^2 \cdot \mathbf{m}^0. \tag{13.4}$$

If the growth part of deformation gradient $\mathbf{F}_g$ is known, the elastic part can be determined from

$$\mathbf{F}_e = \mathbf{F} \cdot \mathbf{F}_g^{-1}. \tag{13.5}$$

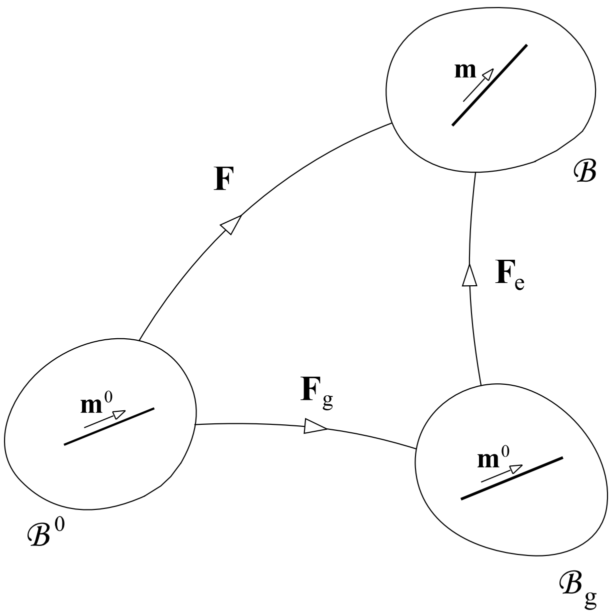

Fig. 2. The fiber orientation of transversely isotropic material in configurations $\mathscr{B}^0$ and $\mathscr{B}_g$ is defined by the unit vector $\mathbf{m}^0$. An infinitesimal fiber segment in the configuration $\mathscr{B}$ is codirectional with the vector $\mathbf{m} = \mathbf{F}_e \cdot \mathbf{m}^0$, obtained from $\mathbf{m}^0$ by elastic stretching and rotation.

The corresponding stress is

$$\boldsymbol{\tau} = 2\mathbf{F}_{\rm e} \cdot \frac{\partial(\rho_{\rm g}^0 \psi)}{\partial \mathbf{C}_{\rm e}} \cdot \mathbf{F}_{\rm e}^{\rm T}, \tag{13.6}$$

i.e.,

$$\boldsymbol{\tau} = 2\left[c_2\mathbf{I} + c_0\mathbf{B}_e + c_1\mathbf{B}_{\rm e}^2 + c_3\mathbf{m}\otimes\mathbf{m} + c_4(\mathbf{B}_{\rm e}\cdot\mathbf{m}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{m}\cdot\mathbf{B}_{\rm e})\right], \tag{13.7}$$

where c_0, c_1, and c_2 are defined by Eq. (12.4), and

$$c_3 = \frac{\partial(\rho_{\rm g}^0\psi)}{\partial M_1}, \quad c_4 = \frac{\partial(\rho_{\rm g}^0\psi)}{\partial M_2}. \tag{13.8}$$

Consequently, the Kirchhoff stress is the elastic part of the rate of deformation $\mathbf{D}_{\rm e}$ is defined by Eq. (11.1), with the elastic moduli specified by Eqs. (11.5) and (11.9). The rectangular components of the elastic moduli tensor $\boldsymbol{\Lambda}_{\rm e}$ are

$$\begin{aligned}\Lambda^{\rm e}_{ijkl} = 4\Big[& a_1\delta_{ij}\delta_{kl} + a_2\delta_{(ik}\delta_{jl)} + a_3\delta_{(ij}C^{\rm e}_{kl)} + a_4C^{\rm e}_{ij}C^{\rm e}_{kl} + a_5\delta_{(ij}C^{\rm e-1}_{kl)} + a_6C^{\rm e}_{(ij}C^{\rm e-1}_{kl)} + a_7C^{\rm e-1}_{(ij}C^{\rm e-1}_{kl)} \\ &+ a_8C^{\rm e-1}_{(ik}C^{\rm e-1}_{jl)} + a_9m^0_{(i}m^0_j\delta_{kl)} + a_{10}m^0_{(i}m^0_k\delta_{jl)} + a_{11}m^0_im^0_jm^0_km^0_l + a_{12}m^0_{(i}m^0_jC^{\rm e}_{kl)} \\ &+ a_{13}m^0_{(i}m^0_jC^{\rm e-1}_{kl)} + a_{14}\delta_{(ij}C_{k\alpha}m^0_\alpha m^0_{l)} + a_{15}m^0_{(i}m^0_jC_{k\alpha}m^0_\alpha m^0_{l)} + a_{16}C^{\rm e}_{(ij}C^{\rm e}_{k\alpha}m^0_\alpha m^0_{l)} \\ &+ a_{17}C^{\rm e-1}_{(ij}C^{\rm e}_{k\alpha}m^0_\alpha m^0_{l)} + a_{18}C^{\rm e}_{(i\alpha}m^0_\alpha m^0_jC^{\rm e}_{k\beta}m^0_\beta m^0_{l)}\Big].\end{aligned} \tag{13.9}$$

The coefficients a_i $(i = 1, 2, \ldots, 18)$ are defined in Appendix A. The symmetrization with respect to i and j, k and l, and ij and kl is used in Eq. (13.9), such that

$$C^{\rm e}_{(i\alpha}m^0_\alpha m^0_{j)} = \tfrac{1}{2}\left(C^{\rm e}_{i\alpha}m^0_\alpha m^0_j + C^{\rm e}_{j\alpha}m^0_\alpha m^0_i\right), \tag{13.10}$$

$$C^{\rm e}_{(ij}C^{\rm e}_{k\alpha}m^0_\alpha m^0_{l)} = \tfrac{1}{2}\left[C^{\rm e}_{ij}C^{\rm e}_{(k\alpha}m^0_\alpha m^0_{l)} + C^{\rm e}_{kl}C^{\rm e}_{(i\alpha}m^0_\alpha m^0_{j)}\right], \tag{13.11}$$

$$C^{\rm e}_{(i\alpha}m^0_\alpha m^0_jC^{\rm e}_{k\beta}m^0_\beta m^0_{l)} = \tfrac{1}{2}C^{\rm e}_{(i\alpha}m^0_\alpha m^0_{j)}C^{\rm e}_{(k\beta}m^0_\beta m^0_{l)}. \tag{13.12}$$

In the case of elastically incompressible material, the Cauchy stress becomes

$$\boldsymbol{\sigma} = -p\mathbf{I} + \frac{2}{J}\left[c_0\mathbf{B}_e + c_1\mathbf{B}_{\rm e}^2 + c_3\mathbf{m}\otimes\mathbf{m} + c_4(\mathbf{B}_{\rm e}\cdot\mathbf{m}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{m}\cdot\mathbf{B}_{\rm e})\right], \tag{13.13}$$

where p is an arbitrary pressure. The rectangular components of the elastic moduli tensor $\boldsymbol{\Lambda}_{\rm e}$ are given by Eq. (13.9) with $a_5 = a_6 = a_7 = a_8 = a_{13} = a_{17} = 0$.

13.1. Elastic strain energy representation

The elastic strain energy function for an incompressible transversely isotropic soft tissue can be taken as in Eq. (12.14), but with Q and q defined by

$$\begin{aligned}Q = {}& \alpha_1(I_C - 3) + \alpha_2(II_C - 3) + \alpha_3(I_C - 3)^2 + \alpha_4(M_1 - 1) + \alpha_5(M_1 - 1)^2 + \alpha_6(I_C - 3)(M_1 - 1) \\ &+ \alpha_7(M_2 - 1),\end{aligned} \tag{13.14}$$

$$\begin{aligned}q = {}& \beta_1(I_C - 3) + \beta_2(II_C - 3) + \beta_3(I_C - 3)^2 + \beta_4(M_1 - 1) + \beta_5(M_1 - 1)^2 + \beta_6(I_C - 3)(M_1 - 1) \\ &+ \beta_7(M_2 - 1).\end{aligned} \tag{13.15}$$

In order that the intermediate configuration is unstressed, we require that $\beta_1 - 2\beta_2 = Jp_0$, and $\beta_4 + 2\beta_7 = 0$. For compressible tissues such as articular cartilage, Almeida and Spilker (1998) proposed for the strain energy function

$$\rho_{\mathrm{g}}^0 \psi = \tfrac{1}{2}\alpha_0 \exp\left(Q - n \ln III_C\right), \tag{13.16}$$

where $\alpha_1 - 2\alpha_2 = n$, and $\alpha_4 + 2\alpha_7 = 0$.

13.2. Transversely isotropic mass growth

We propose the following expression for the growth part of the deformation gradient due to transversely isotropic mass growth,

$$\mathbf{F}_{\mathrm{g}} = \vartheta_{\mathrm{g}}\mathbf{I} + (\eta_{\mathrm{g}} - \vartheta_{\mathrm{g}})\mathbf{m}^0 \otimes \mathbf{m}^0. \tag{13.17}$$

Clearly, $\mathbf{F}_{\mathrm{g}} \cdot \mathbf{m}^0 = \eta_{\mathrm{g}}\mathbf{m}^0$, so that the stretch ratio in the fiber direction is η_{g}, while ϑ_{g} is the stretch ratio in any orthogonal direction. The inverse of the growth deformation tensor is

$$\mathbf{F}_{\mathrm{g}}^{-1} = \frac{1}{\vartheta_{\mathrm{g}}}\mathbf{I} + \left(\frac{1}{\eta_{\mathrm{g}}} - \frac{1}{\vartheta_{\mathrm{g}}}\right)\mathbf{m}^0 \otimes \mathbf{m}^0. \tag{13.18}$$

Consequently, the velocity gradient in the intermediate configuration becomes

$$\dot{\mathbf{F}}_{\mathrm{g}} \cdot \mathbf{F}_{\mathrm{g}}^{-1} = \frac{\dot{\vartheta}_{\mathrm{g}}}{\vartheta_{\mathrm{g}}}\mathbf{I} + \left(\frac{\dot{\eta}_{\mathrm{g}}}{\eta_{\mathrm{g}}} - \frac{\dot{\vartheta}_{\mathrm{g}}}{\vartheta_{\mathrm{g}}}\right)\mathbf{m}^0 \otimes \mathbf{m}^0. \tag{13.19}$$

The corresponding tensor induced by elastic deformation is

$$\mathbf{F}_{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}_{\mathrm{g}} \cdot \mathbf{F}_{\mathrm{g}}^{-1}\right) \cdot \mathbf{F}_{\mathrm{e}}^{-1} = \frac{\dot{\vartheta}_{\mathrm{g}}}{\vartheta_{\mathrm{g}}}\mathbf{I} + \left(\frac{\dot{\eta}_{\mathrm{g}}}{\eta_{\mathrm{g}}} - \frac{\dot{\vartheta}_{\mathrm{g}}}{\vartheta_{\mathrm{g}}}\right)\mathbf{m} \otimes \hat{\mathbf{m}}, \tag{13.20}$$

where

$$\mathbf{m} = \mathbf{F}_{\mathrm{e}} \cdot \mathbf{m}^0, \quad \hat{\mathbf{m}} = \mathbf{m}^0 \cdot \mathbf{F}_{\mathrm{e}}^{-1}. \tag{13.21}$$

Note that $\eta_{\mathrm{g}}\mathbf{m} = \mathbf{F} \cdot \mathbf{m}^0$, $\hat{\mathbf{m}} = \mathbf{B}_{\mathrm{e}}^{-1} \cdot \mathbf{m}$, and $\mathbf{m} \cdot \hat{\mathbf{m}} = 1$. Thus,

$$\mathbf{d}_{\mathrm{g}} = \frac{\dot{\vartheta}_{\mathrm{g}}}{\vartheta_{\mathrm{g}}}\mathbf{I} + \frac{1}{2}\left(\frac{\dot{\eta}_{\mathrm{g}}}{\eta_{\mathrm{g}}} - \frac{\dot{\vartheta}_{\mathrm{g}}}{\vartheta_{\mathrm{g}}}\right)\left(\mathbf{m} \otimes \hat{\mathbf{m}} + \hat{\mathbf{m}} \otimes \mathbf{m}\right), \tag{13.22}$$

and

$$\boldsymbol{\omega}_{\mathrm{g}} = \frac{1}{2}\left(\frac{\dot{\eta}_{\mathrm{g}}}{\eta_{\mathrm{g}}} - \frac{\dot{\vartheta}_{\mathrm{g}}}{\vartheta_{\mathrm{g}}}\right)\left(\mathbf{m} \otimes \hat{\mathbf{m}} - \hat{\mathbf{m}} \otimes \mathbf{m}\right). \tag{13.23}$$

Incorporating the last two expressions in Eq. (11.13) specifies the growth part of the rate of deformation $\mathbf{D}_{\mathrm{g}}$, provided that the evolution equations for the stretch ratios η_{g} and ϑ_{g} are available. Observe also that the substitution of Eq. (13.17) into Eq. (13.5) gives an explicit expression for the elastic part of the deformation gradient

$$\mathbf{F}_{\mathrm{e}} = \frac{1}{\vartheta_{\mathrm{g}}}\mathbf{F} + \left(1 - \frac{\eta_{\mathrm{g}}}{\vartheta_{\mathrm{g}}}\right)\mathbf{m} \otimes \mathbf{m}^0. \tag{13.24}$$

13.3. Evolution equations for stretch ratios

The constitutive formulation is completed by specifying appropriate evolution equations for the stretch ratios ϑ_g and η_g. In the special case when the growth takes place in a density preserving manner ($\rho_g = \rho^0$), we have from Eqs. (9.13) and (13.19)

$$\operatorname{tr}\left(\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}\right) = \frac{\dot{\eta}_g}{\eta_g} + 2\frac{\dot{\vartheta}_g}{\vartheta_g} = \frac{r_g^0}{\rho_g^0}. \tag{13.25}$$

Thus, the rate of the specific mass growth can be expressed in terms of the stretch rates as

$$r_g^0 = \rho_g^0\left(\frac{\dot{\eta}_g}{\eta_g} + 2\frac{\dot{\vartheta}_g}{\vartheta_g}\right). \tag{13.26}$$

Since $r_g^0 = \dot{\rho}_g^0$, the integration of Eq. (13.26), using the initial conditions $\eta_g^0 = 1$, $\vartheta_g^0 = 1$, and $\rho_g^0 = \rho^0$, gives

$$\rho_g^0 = \rho^0 \eta_g \vartheta_g^2. \tag{13.27}$$

The evolution equations for the stretch ratios η_g and ϑ_g must be given in terms of isotropic scalar functions of the stress tensor $\mathbf{T}_e$ and the structural tensor $\mathbf{m}^0 \otimes \mathbf{m}^0$. The past history of the growth is approximately accounted for by including in the list of arguments the current values of the stretch ratios, so that

$$\dot{\eta}_g = f_\eta\left(\eta_g, \vartheta_g, \mathbf{T}_e, \mathbf{m}^0 \otimes \mathbf{m}^0\right), \tag{13.28}$$

$$\dot{\vartheta}_g = f_\vartheta\left(\eta_g, \vartheta_g, \mathbf{T}_e, \mathbf{m}^0 \otimes \mathbf{m}^0\right). \tag{13.29}$$

From the physical point of view, an appealing choice of the arguments of the functions f_η and f_ϑ is indicated below

$$\dot{\eta}_g = f_\eta\left[\eta_g, \vartheta_g, T_m^e, \tfrac{1}{2}\left(\operatorname{tr}\mathbf{T}_e - T_m^e\right)\right], \tag{13.30}$$

$$\dot{\vartheta}_g = f_\vartheta\left[\eta_g, \vartheta_g, T_m^e, \tfrac{1}{2}\left(\operatorname{tr}\mathbf{T}_e - T_m^e\right)\right]. \tag{13.31}$$

The normal component of the Piola–Kirchhoff stress $\mathbf{T}_e$ in the direction of the fiber is $T_m^e = \mathbf{m}^0 \cdot \mathbf{T}_e \cdot \mathbf{m}^0$, while

$$\tfrac{1}{2}\left(\operatorname{tr}\mathbf{T}_e - T_m^e\right) \tag{13.32}$$

represents the average normal stress in the plane perpendicular to the fiber. These two stresses are believed to have a dominant mechanical effect on the transversely isotropic mass growth. Note that the normal stress in the direction of the fiber can be expressed in terms of the Cauchy stress $\boldsymbol{\sigma}$ as

$$\mathbf{m}^0 \cdot \mathbf{T}_e \cdot \mathbf{m}^0 = J_e \hat{\mathbf{m}} \cdot \boldsymbol{\sigma} \cdot \hat{\mathbf{m}}. \tag{13.33}$$

This follows from Eq. (10.6) and the identity $\mathbf{m}^0 \cdot \mathbf{F}_e^{-1} = \mathbf{m} \cdot \mathbf{B}_e^{-1} = \hat{\mathbf{m}}$. In the simplest case the rates of the stretch ratios depend linearly on the stress components, such that

$$\dot{\eta}_g = k_\eta(\eta_g)\left[T_m^e - v_\eta\left(\operatorname{tr}\mathbf{T}_e - T_m^e\right)\right], \tag{13.34}$$

$$\dot{\vartheta}_g = k_\vartheta(\vartheta_g)\left[T_m^e - v_\vartheta\left(\operatorname{tr}\mathbf{T}_e - T_m^e\right)\right], \tag{13.35}$$

where v_η and v_ϑ are the constants, or the functions of η_g and ϑ_g. For example, if $v_\eta = v_\vartheta = v$, the growth-equilibrium state of stress is characterized by $v \operatorname{tr}\mathbf{T}_e = (1+v)T_m^e$ (see also Taber and Perucchio, 2000).

To prevent an unlimited growth at an arbitrary non-zero state of stress, we propose that during the mass growth

$$k_\eta(\eta_{\rm g}) = k^+_{\eta 0}\left(\frac{\eta^+_{\rm g} - \eta_{\rm g}}{\eta^+_{\rm g} - 1}\right)^{m^+_\eta}, \quad T^{\rm e}_{\rm m} > \frac{\nu_\eta}{1+\nu_\eta}\,{\rm tr}\,\mathbf{T}_{\rm e}, \tag{13.36}$$

where $\eta^+_{\rm g} > 1$ is the limiting value of the stretch ratio $\eta_{\rm g}$ that can be attained by mass growth, and $k^+_{\eta 0}$ and m^+_η are the material parameters. In the case of the mass resorption, the corresponding expression is

$$k_\eta(\eta_{\rm g}) = k^-_{\eta 0}\left(\frac{\eta_{\rm g} - \eta^-_{\rm g}}{1 - \eta^-_{\rm g}}\right)^{m^-_\eta}, \quad T^{\rm e}_{\rm m} < \frac{\nu_\eta}{1+\nu_\eta}\,{\rm tr}\,\mathbf{T}_{\rm e}. \tag{13.37}$$

Similar expressions hold for the function k_ϑ in Eq. (13.35). Other than power-type expressions can be used, as required by reproducing particular experimental data.

14. Orthotropic materials

Suppose that the material in the initial configuration $\mathscr{B}^0$ is characterized by an orthogonal network of fibers as orthotropic. Let the unit vectors $\mathbf{m}^0$, $\mathbf{n}^0$, and $\mathbf{m}^0 \times \mathbf{n}^0$ specify the principal axes of orthotropy in both initial and intermediate configuration (e.g., Boehler, 1987; Lubarda, 2002). The intermediate configuration is defined to have the same fiber orientations relative to the fixed frame of reference as does the initial configurations. It is assumed that the orthotropic symmetry remains preserved during the mass growth and that the fibers are imbedded in the material. This is ensured by defining $\mathbf{F}_{\rm g}$ such that $\mathbf{m}^0$ and $\mathbf{n}^0$ are its eigendirections, i.e.,

$$\mathbf{F}_{\rm g}\cdot\mathbf{m}^0 = \eta_{\rm g}\mathbf{m}^0, \quad \mathbf{F}_{\rm g}\cdot\mathbf{n}^0 = \zeta_{\rm g}\mathbf{n}^0, \quad \mathbf{F}_{\rm g}\cdot\left(\mathbf{m}^0\times\mathbf{n}^0\right) = \vartheta_{\rm g}\left(\mathbf{m}^0\times\mathbf{n}^0\right). \tag{14.1}$$

The infinitesimal fiber segments in the configuration $\mathscr{B}$ are obtained from those in the intermediate configuration by elastic embedding. For example, $\mathbf{m} = \mathbf{F}_{\rm e}\cdot\mathbf{m}^0$ and $\mathbf{n} = \mathbf{F}_{\rm e}\cdot\mathbf{n}^0$. The elastic strain energy per unit initial volume can be written as an isotropic function of the elastic deformation tensor $\mathbf{C}_{\rm e}$, and the structural tensors $\mathbf{m}^0\otimes\mathbf{m}^0$ and $\mathbf{n}^0\otimes\mathbf{n}^0$,

$$\rho^0_{\rm g}\psi = \rho^0_{\rm g}\psi\left(\mathbf{C}_{\rm e}, \mathbf{m}^0\otimes\mathbf{m}^0, \mathbf{n}^0\otimes\mathbf{n}^0, \rho^0_{\rm g}\right). \tag{14.2}$$

More specifically, the function ψ can be expressed in terms of individual and joint invariants of $\mathbf{C}_{\rm e}$, $\mathbf{m}^0$ and $\mathbf{n}^0$, such that

$$\psi = \psi(I_C, II_C, III_C, M_1, M_2, N_1, N_2, \rho^0_{\rm g}). \tag{14.3}$$

The joint invariants are

$$M_1 = \mathbf{m}^0\cdot\mathbf{C}_{\rm e}\cdot\mathbf{m}^0, \quad M_2 = \mathbf{m}^0\cdot\mathbf{C}^2_{\rm e}\cdot\mathbf{m}^0, \quad N_1 = \mathbf{n}^0\cdot\mathbf{C}_{\rm e}\cdot\mathbf{n}^0, \quad N_2 = \mathbf{n}^0\cdot\mathbf{C}^2_{\rm e}\cdot\mathbf{n}^0. \tag{14.4}$$

If the growth part of deformation gradient $\mathbf{F}_{\rm g}$ is known, the elastic part can be determined from $\mathbf{F}_{\rm e} = \mathbf{F}\cdot\mathbf{F}^{-1}_{\rm g}$. The corresponding stress is determined from Eq. (13.6). The gradient of the elastic strain energy is

$$\begin{aligned}\frac{\partial(\rho^0_{\rm g}\psi)}{\partial\mathbf{C}_{\rm e}} = {} & c_0\mathbf{I} + c_1\mathbf{C}_e + c_2\mathbf{C}^{-1}_{\rm e} + c_3\mathbf{m}^0\otimes\mathbf{m}^0 + c_4\left(\mathbf{C}_{\rm e}\cdot\mathbf{m}^0\otimes\mathbf{m}^0 + \mathbf{m}^0\otimes\mathbf{m}^0\cdot\mathbf{C}_{\rm e}\right) + c_5\mathbf{n}^0\otimes\mathbf{n}^0 \\ & + c_6\left(\mathbf{C}_{\rm e}\cdot\mathbf{n}^0\otimes\mathbf{n}^0 + \mathbf{n}^0\otimes\mathbf{n}^0\cdot\mathbf{C}_{\rm e}\right),\end{aligned} \tag{14.5}$$

where c_0, c_1, and c_2 are defined by Eq. (12.4), and

$$c_3 = \frac{\partial(\rho_g^0\psi)}{\partial M_1}, \quad c_4 = \frac{\partial(\rho_g^0\psi)}{\partial M_2}, \quad c_5 = \frac{\partial(\rho_g^0\psi)}{\partial N_1}, \quad c_6 = \frac{\partial(\rho_g^0\psi)}{\partial N_2}. \tag{14.6}$$

Consequently, the Kirchhoff stress is

$$\begin{aligned}\boldsymbol{\tau} = 2\big[&c_2\mathbf{I} + c_0\mathbf{B}_e + c_1\mathbf{B}_e^2 + c_3\mathbf{m}\otimes\mathbf{m} + c_4(\mathbf{B}_e\cdot\mathbf{m}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{m}\cdot\mathbf{B}_e) + c_5\mathbf{n}\otimes\mathbf{n}\\ &+ c_6(\mathbf{B}_e\cdot\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{n}\cdot\mathbf{B}_e)\big].\end{aligned} \tag{14.7}$$

The elastic part of the rate of deformation $\mathbf{D}_e$ is defined by Eq. (11.1), with the elastic moduli specified by Eqs. (11.5) and (11.9). The rectangular components of the elastic moduli tensor $\boldsymbol{\Lambda}_e$ are

$$\begin{aligned}\Lambda^e_{ijkl} = 4\Big[&a_1\delta_{ij}\delta_{kl} + a_2\delta_{(ik}\delta_{jl)} + a_3\delta_{(ij}C^e_{kl)} + a_4C^e_{ij}C^e_{kl} + a_5\delta_{(ij}C^{e-1}_{kl)} + a_6C^e_{(ij}C^{e-1}_{kl)} + a_7C^{e-1}_{(ij}C^{e-1}_{kl)}\\ &+ a_8C^{e-1}_{(ik}C^{e-1}_{jl)} + a_9m^0_{(i}m^0_j\delta_{kl)} + a_{10}m^0_{(i}m^0_k\delta_{jl)} + a_{11}m^0_im^0_jm^0_km^0_l + a_{12}m^0_{(i}m^0_jC^e_{kl)}\\ &+ a_{13}m^0_{(i}m^0_jC^{e-1}_{kl)} + a_{14}\delta_{(ij}C_{k\alpha}m^0_\alpha m^0_{l)} + a_{15}m^0_{(i}m^0_jC_{k\alpha}m^0_\alpha m^0_{l)} + a_{16}C^e_{(ij}C^e_{k\alpha}m^0_\alpha m^0_{l)}\\ &+ a_{17}C^{e-1}_{(ij}C^e_{k\alpha}m^0_\alpha m^0_{l)} + a_{18}C^e_{(i\alpha}m^0_\alpha m^0_jC^e_{k\beta}m^0_\beta m^0_{l)} + a_{19}n^0_{(i}n^0_j\delta_{kl)} + a_{20}n^0_{(i}n^0_k\delta_{jl)} + a_{21}n^0_in^0_jn^0_kn^0_l\\ &+ a_{22}n^0_{(i}n^0_jC^e_{kl)} + a_{23}n^0_{(i}n^0_jC^{e-1}_{kl)} + a_{24}\delta_{(ij}C_{k\alpha}n^0_\alpha n^0_{l)} + a_{25}n^0_{(i}n^0_jC_{k\alpha}n^0_\alpha n^0_{l)} + a_{26}C^e_{(ij}C^e_{k\alpha}n^0_\alpha n^0_{l)}\\ &+ a_{27}C^{e-1}_{(ij}C^e_{k\alpha}n^0_\alpha n^0_{l)} + a_{28}C^e_{(i\alpha}n^0_\alpha n^0_jC^e_{k\beta}n^0_\beta n^0_{l)}\Big].\end{aligned} \tag{14.8}$$

The coefficients a_i $(i = 1, 2, \ldots, 28)$ are defined in Appendix A.

In the case of elastically incompressible material, the Cauchy stress becomes

$$\begin{aligned}\boldsymbol{\sigma} = -p\mathbf{I} + \frac{2}{J}\big[&c_0\mathbf{B}_e + c_1\mathbf{B}_e^2 + c_3\mathbf{m}\otimes\mathbf{m} + c_4(\mathbf{B}_e\cdot\mathbf{m}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{m}\cdot\mathbf{B}_e)\\ &+ c_5\mathbf{n}\otimes\mathbf{n} + c_6(\mathbf{B}_e\cdot\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{n}\cdot\mathbf{B}_e)\big],\end{aligned} \tag{14.9}$$

where p is an arbitrary pressure. The rectangular components of the elastic moduli tensor $\boldsymbol{\Lambda}_e$ are given by Eq. (14.8) with $a_5 = a_6 = a_7 = a_8 = a_{13} = a_{17} = a_{23} = a_{27} = 0$.

14.1. Elastic strain energy representation

The elastic strain energy function for an incompressible orthotropic soft tissue can be taken as in Eq. (12.14), but with Q and q defined by

$$\begin{aligned}Q = {}&\alpha_1(I_C - 3) + \alpha_2(II_C - 3) + \alpha_3(I_C - 3)^2 + \alpha_4(M_1 - 1) + \alpha_5(M_1 - 1)^2 + \alpha_6(I_C - 3)(M_1 - 1)\\ &+ \alpha_7(M_2 - 1) + \alpha_8(N_1 - 1) + \alpha_9(N_1 - 1)^2 + \alpha_{10}(I_C - 3)(N_1 - 1) + \alpha_{11}(N_2 - 1)\\ &+ \alpha_{12}(M_1 - 1)(N_1 - 1),\end{aligned} \tag{14.10}$$

$$\begin{aligned}q = {}&\beta_1(I_C - 3) + \beta_2(II_C - 3) + \beta_3(I_C - 3)^2 + \beta_4(M_1 - 1) + \beta_5(M_1 - 1)^2 + \beta_6(I_C - 3)(M_1 - 1)\\ &+ \beta_7(M_2 - 1) + \beta_8(N_1 - 1) + \beta_9(N_1 - 1)^2 + \beta_{10}(I_C - 3)(N_1 - 1) + \beta_{11}(N_2 - 1)\\ &+ \beta_{12}(M_1 - 1)(N_1 - 1).\end{aligned} \tag{14.11}$$

In order that the intermediate configuration is unstressed, we require that $\beta_1 - 2\beta_2 = Jp_0$, $\beta_4 + 2\beta_7 = 0$, and $\beta_8 + 2\beta_{11} = 0$. Holzapfel and Weizsäcker (1998) used a simpler structure of the elastic strain energy, a variant of which is

$$\rho_g^0 \psi = \tfrac{1}{2}\alpha_0(\exp Q - 1) + \tfrac{1}{2}\beta_1(I_C - 3) - \tfrac{1}{2}p(III_C - 1), \tag{14.12}$$

where $\alpha_0(\alpha_1 - 2\alpha_2) + \beta_1 = Jp_0$, and $\alpha_4 + 2\alpha_7 = 0$.

14.2. Orthotropic mass growth

In view of the previous discussion, the following expression for the growth part of the deformation gradient due to orthotropic mass growth suggests itself

$$\mathbf{F}_g = \vartheta_g \mathbf{I} + (\eta_g - \vartheta_g)\mathbf{m}^0 \otimes \mathbf{m}^0 + (\zeta_g - \vartheta_g)\mathbf{n}^0 \otimes \mathbf{n}^0, \tag{14.13}$$

where η_g and ζ_g are the stretch ratios in the directions $\mathbf{m}^0$ and $\mathbf{n}^0$, while ϑ_g is the stretch ratio in the direction $\mathbf{m}^0 \times \mathbf{n}^0$. The inverse of the growth deformation tensor is

$$\mathbf{F}_g^{-1} = \frac{1}{\vartheta_g}\mathbf{I} + \left(\frac{1}{\eta_g} - \frac{1}{\vartheta_g}\right)\mathbf{m}^0 \otimes \mathbf{m}^0 + \left(\frac{1}{\zeta_g} - \frac{1}{\vartheta_g}\right)\mathbf{n}^0 \otimes \mathbf{n}^0. \tag{14.14}$$

The velocity gradient in the intermediate configuration can be written as

$$\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1} = \frac{\dot{\vartheta}_g}{\vartheta_g}\mathbf{I} + \left(\frac{\dot{\eta}_g}{\eta_g} - \frac{\dot{\vartheta}_g}{\vartheta_g}\right)\mathbf{m}^0 \otimes \mathbf{m}^0 + \left(\frac{\dot{\zeta}_g}{\zeta_g} - \frac{\dot{\vartheta}_g}{\vartheta_g}\right)\mathbf{n}^0 \otimes \mathbf{n}^0. \tag{14.15}$$

The corresponding tensor induced by elastic deformation is

$$\mathbf{F}_e \cdot \left(\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}\right) \cdot \mathbf{F}_e^{-1} = \frac{\dot{\vartheta}_g}{\vartheta_g}\mathbf{I} + \left(\frac{\dot{\eta}_g}{\eta_g} - \frac{\dot{\vartheta}_g}{\vartheta_g}\right)\mathbf{m} \otimes \hat{\mathbf{m}} + \left(\frac{\dot{\zeta}_g}{\zeta_g} - \frac{\dot{\vartheta}_g}{\vartheta_g}\right)\mathbf{n} \otimes \hat{\mathbf{n}}, \tag{14.16}$$

where $\mathbf{m}$ and $\hat{\mathbf{m}}$ are defined by Eq. (13.21), while

$$\mathbf{n} = \mathbf{F}_e \cdot \mathbf{n}^0, \quad \hat{\mathbf{n}} = \mathbf{n}^0 \cdot \mathbf{F}_e^{-1}. \tag{14.17}$$

It is noted that

$$\eta_g \mathbf{m} = \mathbf{F} \cdot \mathbf{m}^0, \quad \zeta_g \mathbf{n} = \mathbf{F} \cdot \mathbf{n}^0, \tag{14.18}$$

$$\hat{\mathbf{m}} = \mathbf{B}_e^{-1} \cdot \mathbf{m}, \quad \hat{\mathbf{n}} = \mathbf{B}_e^{-1} \cdot \mathbf{n}. \tag{14.19}$$

The properties $\mathbf{m} \cdot \hat{\mathbf{m}} = 1$ and $\mathbf{n} \cdot \hat{\mathbf{n}} = 1$ are easily verified. Thus, we have

$$\mathbf{d}_g = \frac{\dot{\vartheta}_g}{\vartheta_g}\mathbf{I} + \frac{1}{2}\left(\frac{\dot{\eta}_g}{\eta_g} - \frac{\dot{\vartheta}_g}{\vartheta_g}\right)\left(\mathbf{m} \otimes \hat{\mathbf{m}} + \hat{\mathbf{m}} \otimes \mathbf{m}\right) + \frac{1}{2}\left(\frac{\dot{\zeta}_g}{\zeta_g} - \frac{\dot{\vartheta}_g}{\vartheta_g}\right)(\mathbf{n} \otimes \hat{\mathbf{n}} + \hat{\mathbf{n}} \otimes \mathbf{n}), \tag{14.20}$$

and

$$\boldsymbol{\omega}_g = \frac{1}{2}\left(\frac{\dot{\eta}_g}{\eta_g} - \frac{\dot{\vartheta}_g}{\vartheta_g}\right)\left(\mathbf{m} \otimes \hat{\mathbf{m}} - \hat{\mathbf{m}} \otimes \mathbf{m}\right) + \frac{1}{2}\left(\frac{\dot{\zeta}_g}{\zeta_g} - \frac{\dot{\vartheta}_g}{\vartheta_g}\right)(\mathbf{n} \otimes \hat{\mathbf{n}} - \hat{\mathbf{n}} \otimes \mathbf{n}). \tag{14.21}$$

Incorporating the last two expressions into Eq. (11.13) specifies the growth part of the rate of deformation $\mathbf{D}_g$, provided that the evolution equations for the stretch ratios η_g, ζ_g, and ϑ_g are available. Observe also an explicit expression for the elastic part of the deformation gradient

$$\mathbf{F}_e = \frac{1}{\vartheta_g}\mathbf{F} + \left(1 - \frac{\eta_g}{\vartheta_g}\right)\mathbf{m} \otimes \mathbf{m}^0 + \left(1 - \frac{\zeta_g}{\vartheta_g}\right)\mathbf{n} \otimes \mathbf{n}^0. \tag{14.22}$$

14.3. Evolution equations for stretch ratios

The constitutive formulation is completed by specifying the appropriate evolution equations for the stretch ratios η_g, ζ_g, and ϑ_g. In the special case when the growth takes place in a density preserving manner ($\rho_g = \rho^0$), we have from Eqs. (9.13) and (14.15)

$$\mathrm{tr}(\dot{\mathbf{F}}_g \cdot \mathbf{F}_g^{-1}) = \frac{\dot{\eta}_g}{\eta_g} + \frac{\dot{\zeta}_g}{\zeta_g} + \frac{\dot{\vartheta}_g}{\vartheta_g} = \frac{r_g^0}{\rho_g^0}. \tag{14.23}$$

Thus, the rate of the specific mass growth can be expressed in terms of the stretch rates as

$$r_g^0 = \rho_g^0 \left(\frac{\dot{\eta}_g}{\eta_g} + \frac{\dot{\zeta}_g}{\zeta_g} + \frac{\dot{\vartheta}_g}{\vartheta_g} \right). \tag{14.24}$$

Upon integration, using the initial conditions $\eta_g^0 = 1$, $\zeta_g^0 = 1$, $\vartheta_g^0 = 1$, and $\rho_g^0 = \rho^0$, there follows

$$\rho_g^0 = \rho^0 \eta_g \zeta_g \vartheta_g. \tag{14.25}$$

The evolution equations for the stretch ratios η_g, ζ_g, and ϑ_g are given in terms of isotropic functions of the stress tensor $\mathbf{T}_e$, and the structural tensors $\mathbf{m}^0 \otimes \mathbf{m}^0$ and $\mathbf{n}^0 \otimes \mathbf{n}^0$. The past history of the growth is approximately accounted for by including in the list of arguments the current values of the stretch ratios, so that

$$\dot{\eta}_g = f_\eta\left(\eta_g, \zeta_g, \vartheta_g, \mathbf{T}_e, \mathbf{m}^0 \otimes \mathbf{m}^0, \mathbf{n}^0 \otimes \mathbf{n}^0\right), \tag{14.26}$$

$$\dot{\zeta}_g = f_\zeta\left(\eta_g, \zeta_g, \vartheta_g, \mathbf{T}_e, \mathbf{m}^0 \otimes \mathbf{m}^0, \mathbf{n}^0 \otimes \mathbf{n}^0\right), \tag{14.27}$$

$$\dot{\vartheta}_g = f_\vartheta\left(\eta_g, \zeta_g, \vartheta_g, \mathbf{T}_e, \mathbf{m}^0 \otimes \mathbf{m}^0, \mathbf{n}^0 \otimes \mathbf{n}^0\right). \tag{14.28}$$

An appealing choice of the arguments of the functions f_η, f_ζ, and f_ϑ is as indicated below

$$\dot{\eta}_g = f_\eta\left[\eta_g, \zeta_g, \vartheta_g, T_m^e, \tfrac{1}{2}\left(\mathrm{tr}\,\mathbf{T}_e - T_m^e\right)\right], \tag{14.29}$$

$$\dot{\zeta}_g = f_\zeta\left[\eta_g, \zeta_g, \vartheta_g, T_n^e, \tfrac{1}{2}\left(\mathrm{tr}\,\mathbf{T}_e - T_n^e\right)\right], \tag{14.30}$$

$$\dot{\vartheta}_g = f_\vartheta\left[\eta_g, \zeta_g, \vartheta_g, T_p^e, \tfrac{1}{2}\left(\mathrm{tr}\,\mathbf{T}_e - T_p^e\right)\right]. \tag{14.31}$$

The normal components of the Piola–Kirchhoff stress in the intermediate configuration codirectional with $\mathbf{m}^0$, $\mathbf{n}^0$, and $\mathbf{p}^0 = \mathbf{m}^0 \times \mathbf{n}^0$ are $T_m^e = \mathbf{m}^0 \cdot \mathbf{T}_e \cdot \mathbf{m}^0$, $T_n^e = \mathbf{n}^0 \cdot \mathbf{T}_e \cdot \mathbf{n}^0$, and $T_p^e = \mathbf{p}^0 \cdot \mathbf{T}_e \cdot \mathbf{p}^0$, while

$$\tfrac{1}{2}\left(\mathrm{tr}\,\mathbf{T}_e - T_m^e\right), \quad \tfrac{1}{2}\left(\mathrm{tr}\,\mathbf{T}_e - T_n^e\right), \quad \tfrac{1}{2}\left(\mathrm{tr}\,\mathbf{T}_e - T_p^e\right) \tag{14.32}$$

are the average normal stresses in the planes perpendicular to $\mathbf{m}^0$, $\mathbf{n}^0$, and $\mathbf{p}^0$, respectively. These stress components are believed to have a dominant mechanical effect on the orthotropic mass growth. It is also noted that the Piola–Kirhhoff normal stress components can be expressed in terms of the Cauchy stress $\boldsymbol{\sigma}$ as

$$\mathbf{m}^0 \cdot \mathbf{T}_e \cdot \mathbf{m}^0 = J_e \hat{\mathbf{m}} \cdot \boldsymbol{\sigma} \cdot \hat{\mathbf{m}}, \quad \mathbf{n}^0 \cdot \mathbf{T}_e \cdot \mathbf{n}^0 = J_e \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}, \quad \mathbf{p}^0 \cdot \mathbf{T}_e \cdot \mathbf{p}^0 = J_e \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \cdot \hat{\mathbf{p}}. \tag{14.33}$$

This follows from Eq. (10.6) and the identities

$$\mathbf{m}^0 \cdot \mathbf{F}_e^{-1} = \mathbf{m} \cdot \mathbf{B}_e^{-1} = \hat{\mathbf{m}}, \quad \mathbf{n}^0 \cdot \mathbf{F}_e^{-1} = \mathbf{n} \cdot \mathbf{B}_e^{-1} = \hat{\mathbf{n}}, \quad \mathbf{p}^0 \cdot \mathbf{F}_e^{-1} = \mathbf{p} \cdot \mathbf{B}_e^{-1} = \hat{\mathbf{p}}. \tag{14.34}$$

In the simplest case the rates of the stretch ratios depend linearly on the stress components, such that

$$\dot{\eta}_{\mathrm{g}} = k_\eta(\eta_{\mathrm{g}})\left[T^{\mathrm{e}}_{\mathrm{m}} - v_\eta\left(\mathrm{tr}\,\mathbf{T}_{\mathrm{e}} - T^{\mathrm{e}}_{\mathrm{m}}\right)\right], \tag{14.35}$$

$$\dot{\zeta}_{\mathrm{g}} = k_\zeta(\zeta_{\mathrm{g}})\left[T^{\mathrm{e}}_{\mathrm{n}} - v_\zeta\left(\mathrm{tr}\,\mathbf{T}_{\mathrm{e}} - T^{\mathrm{e}}_{\mathrm{n}}\right)\right], \tag{14.36}$$

$$\dot{\vartheta}_{\mathrm{g}} = k_\vartheta(\vartheta_{\mathrm{g}})[T^{\mathrm{e}}_{\mathrm{p}} - v_\vartheta(\mathrm{tr}\,\mathbf{T}_{\mathrm{e}} - T^{\mathrm{e}}_{\mathrm{p}})], \tag{14.37}$$

where v_η, v_ζ, and v_ϑ are the constants, or appropriate functions of η_{g}, ζ_{g}, and ϑ_{g}. Similar evolution equations for the stretch ratios with linear dependence on stress were used by Taber and Eggers (1996) in the analysis of stress-modulated growth in the aorta with cylindrically orthotropic material properties. To prevent an unlimited growth at an arbitrary non-zero state of stress, we propose that during the mass growth

$$k_\eta(\eta_{\mathrm{g}}) = k^+_{\eta 0}\left(\frac{\eta^+_{\mathrm{g}} - \eta_{\mathrm{g}}}{\eta^+_{\mathrm{g}} - 1}\right)^{m^+_\eta}, \quad T^{\mathrm{e}}_{\mathrm{m}} > \frac{v_\eta}{1 + v_\eta}\mathrm{tr}\,\mathbf{T}_{\mathrm{e}}, \tag{14.38}$$

where $\eta^+_{\mathrm{g}} > 1$ is the limiting value of the stretch ratio η_{g} that can be attained by mass growth, and $k^+_{\eta 0}$ and m^+_η are the material parameters. In the case of the mass resorption, the corresponding expression is

$$k_\eta(\eta_{\mathrm{g}}) = k^-_{\eta 0}\left(\frac{\eta_{\mathrm{g}} - \eta^-_{\mathrm{g}}}{1 - \eta^-_{\mathrm{g}}}\right)^{m^-_\eta}, \quad T^{\mathrm{e}}_{\mathrm{m}} < \frac{v_\eta}{1 + v_\eta}\mathrm{tr}\,\mathbf{T}_{\mathrm{e}}. \tag{14.39}$$

Similar expressions hold for the functions k_ζ and k_ϑ and in Eqs. (14.36) and (14.37).

15. Orthotropic material with transversely isotropic mass growth

Some tissues are characterized by a regular (alternating) sequence of two families of unidirectionally fiber-reinforced thin sheets. One family of sheets has fibers parallel to the unit vector $\mathbf{a}^0$, and the other parallel to the unit vector $\mathbf{b}^0$. The two sets of fibers are not necessarily orthogonal, but are inclined at an angle 2Φ, such that

$$\mathbf{a}^0 \cdot \mathbf{b}^0 = \cos 2\Phi. \tag{15.1}$$

All fibers are assumed to be mechanically and otherwise equivalent, so that the bisectors of the angles between $\mathbf{a}^0$ and $\mathbf{b}^0$ are the planes of elastic symmetry. The material is thus elastically orthotropic, with the principal axes of orthotropy given by (Fig. 3)

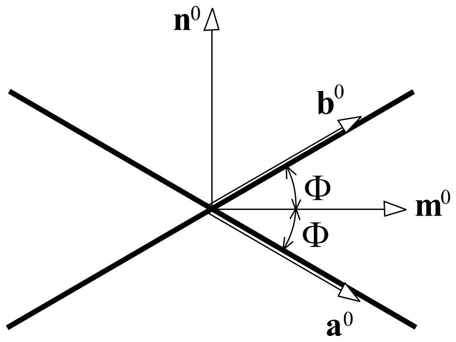

Fig. 3. Two sets of fibers parallel to unit vectors $\mathbf{a}^0$ and $\mathbf{b}^0$ are at an angle 2Φ. For mechanically equivalent fibers the material is orthotropic with principal axes of orthotropy in the directions $\mathbf{m}^0$, $\mathbf{n}^0$, and $\mathbf{m}^0 \times \mathbf{n}^0$.

$$\mathbf{m}^0 = \frac{1}{2\cos\Phi}(\mathbf{a}^0 + \mathbf{b}^0), \quad \mathbf{n}^0 = \frac{1}{2\sin\Phi}(\mathbf{b}^0 - \mathbf{a}^0), \quad \mathbf{m}^0 \times \mathbf{n}^0. \tag{15.2}$$

The elastic strain energy can then be expressed as a function of the following invariants (e.g., Spencer, 1982)

$$\begin{aligned} &I_C, \quad II_C, \quad III_C, \quad \cos^2 2\Phi, \\ &K_1 = \mathbf{a}^0 \cdot \mathbf{C}_e \cdot \mathbf{a}^0 + \mathbf{b}^0 \cdot \mathbf{C}_e \cdot \mathbf{b}^0, \quad K_2 = \left(\mathbf{a}^0 \cdot \mathbf{C}_e \cdot \mathbf{a}^0\right)\left(\mathbf{b}^0 \cdot \mathbf{C}_e \cdot \mathbf{b}^0\right), \\ &K_3 = \mathbf{a}^0 \cdot \mathbf{C}_e^2 \cdot \mathbf{a}^0 + \mathbf{b}^0 \cdot \mathbf{C}_e^2 \cdot \mathbf{b}^0, \quad K_4 = \left(\mathbf{a}^0 \cdot \mathbf{C}_e \cdot \mathbf{b}^0\right)\cos 2\Phi. \end{aligned} \tag{15.3}$$

The gradient of the elastic strain energy is

$$\begin{aligned} \frac{\partial(\rho_g^0\psi)}{\partial \mathbf{C}_e} = {} & c_0\mathbf{I} + c_1\mathbf{C}_e + c_2\mathbf{C}_e^{-1} + d_1(\mathbf{a}^0 \otimes \mathbf{a}^0 + \mathbf{b}^0 \otimes \mathbf{b}^0) + d_2\big[(\mathbf{a}^0 \otimes \mathbf{a}^0)\left(\mathbf{b}^0 \cdot \mathbf{C}_e \cdot \mathbf{b}^0\right) \\ & + (\mathbf{b}^0 \otimes \mathbf{b}^0)\left(\mathbf{a}^0 \cdot \mathbf{C}_e \cdot \mathbf{a}^0\right)\big] + d_3\left[\mathbf{C}_e \cdot (\mathbf{a}^0 \otimes \mathbf{a}^0 + \mathbf{b}^0 \otimes \mathbf{b}^0) + (\mathbf{a}^0 \otimes \mathbf{a}^0 + \mathbf{b}^0 \otimes \mathbf{b}^0) \cdot \mathbf{C}_e\right] \\ & + \tfrac{1}{2}d_4(\mathbf{a}^0 \otimes \mathbf{b}^0 + \mathbf{b}^0 \otimes \mathbf{a}^0)\cos 2\Phi, \end{aligned} \tag{15.4}$$

with c_0, c_1, and c_2 defined by Eq. (12.4), and

$$d_r = \frac{\partial(\rho_g^0\psi)}{\partial K_r}, \quad r = 1, 2, 3, 4. \tag{15.5}$$

Consequently, the Kirchhoff stress is

$$\begin{aligned} \boldsymbol{\tau} = 2\big\{ & c_2\mathbf{I} + c_0\mathbf{B}_e + c_1\mathbf{B}_e^2 + d_1(\mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}) + d_2[(\mathbf{a} \otimes \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) + (\mathbf{b} \otimes \mathbf{b})(\mathbf{a} \cdot \mathbf{a})] \\ & + d_3[\mathbf{B}_e \cdot (\mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}) + (\mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}) \cdot \mathbf{B}_e] + \tfrac{1}{2}d_4(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})\cos 2\Phi\big\}, \end{aligned} \tag{15.6}$$

where $\mathbf{a} = \mathbf{F}_e \cdot \mathbf{a}^0$ and $\mathbf{b} = \mathbf{F}_e \cdot \mathbf{b}^0$. The rectangular components of the elastic moduli tensor $\boldsymbol{\Lambda}_e$ are

$$\begin{aligned} \Lambda_{ijkl}^e = 4\Bigg\{ & a_1\delta_{ij}\delta_{kl} + a_2\delta_{(ik}\delta_{jl)} + a_3\delta_{(ij}C_{kl)}^e + a_4C_{ij}^eC_{kl}^e + a_5\delta_{(ij}C_{kl)}^{e-1} + a_6C_{(ij}^eC_{kl)}^{e-1} + a_7C_{(ij}^{e-1}C_{kl)}^{e-1} \\ & + a_8C_{(ik}^{e-1}C_{jl)}^{e-1} + \frac{\partial d_1}{\partial C_{(ij}^e}\left[a_k^0a_{l)}^0 + b_k^0b_{l)}^0\right] + \frac{\partial d_2}{\partial C_{(ij}^e}\left[a_k^0a_{l)}^0C_b^e + b_k^0b_{l)}^0C_a^e\right] + 2\frac{\partial d_3}{\partial C_{(ij}^e}C_{k\alpha}^e\left[a_\alpha^0a_{l)}^0 + b_\alpha^0b_{l)}^0\right] \\ & + 2\frac{\partial d_4}{\partial C_{(ij}^e}a_k^0b_{l)}^0\cos 2\Phi + 2d_2a_{(i}^0a_j^0b_k^0b_{l)}^0 + 2d_3\delta_{(ik}\left[a_j^0a_{l)}^0 + b_j^0b_{l)}^0\right]\Bigg\}, \end{aligned} \tag{15.7}$$

where $C_a^e = \mathbf{a}^0 \cdot \mathbf{C}_e \cdot \mathbf{a}^0$ and $C_b^e = \mathbf{b}^0 \cdot \mathbf{C}_e \cdot \mathbf{b}^0$. The symmetrization is defined such that, for example,

$$\frac{\partial d_4}{\partial C_{(ij}^e}a_k^0b_{l)}^0 = \frac{1}{4}\left[\frac{\partial d_4}{\partial C_{ij}^e}\left(a_k^0b_l^0 + b_k^0a_l^0\right) + \frac{\partial d_4}{\partial C_{kl}^e}\left(a_i^0b_j^0 + b_i^0a_j^0\right)\right]. \tag{15.8}$$

The explicit expressions for the gradients of d_r with respect to $\mathbf{C}_e$ are listed in Appendix A.

In the case of elastically incompressible material, the Cauchy stress becomes

$$\begin{aligned} \boldsymbol{\sigma} = -p\mathbf{I} + \frac{2}{J}\Bigg\{ & c_2\mathbf{I} + c_0\mathbf{B}_e + c_1\mathbf{B}_e^2 + d_1(\mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}) + d_2[(\mathbf{a} \otimes \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) + (\mathbf{b} \otimes \mathbf{b})(\mathbf{a} \cdot \mathbf{a})] \\ & + d_3[\mathbf{B}_e \cdot (\mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}) + (\mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}) \cdot \mathbf{B}_e] + \frac{1}{2}d_4(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})\cos 2\Phi\Bigg\}, \end{aligned} \tag{15.9}$$

where p is an arbitrary pressure. The rectangular components of the elastic moduli tensor $\boldsymbol{\Lambda}_e$ are given by Eq. (15.7) with $a_5 = a_6 = a_7 = a_8 = 0$, and $b_{13} = b_{23} = b_{33} = 0$ in Eq. (A.21) of Appendix A.

15.1. Elastic strain energy representation

The elastic strain energy function for an incompressible orthotropic soft tissue can be taken as in Eq. (12.14), but with Q and q defined by

$$\begin{aligned} Q = {} & \alpha_1(I_C-3)+\alpha_2(II_C-3)+\alpha_3(I_C-3)^2+\alpha_4(K_1-2)+\alpha_5(K_1-2)^2+\alpha_6(I_C-3)(K_1-2) \\ & +\alpha_7(K_2-1)+\alpha_8(K_3-2)+\alpha_9\left(K_4-\cos^2 2\Phi\right)+\alpha_{10}\left(K_4-\cos^2 2\Phi\right)^2+\alpha_{11}(I_C-3)\left(K_4-\cos^2 2\Phi\right) \\ & +\alpha_{12}(K_1-2)\left(K_4-\cos^2 2\Phi\right), \end{aligned} \tag{15.10}$$

$$\begin{aligned} q = {} & \beta_1(I_C-3)+\beta_2(II_C-3)+\beta_3(I_C-3)^2+\beta_4(K_1-2)+\beta_5(K_1-2)^2+\beta_6(I_C-3)(K_1-2) \\ & +\beta_7(K_2-1)+\beta_8(K_3-2)+\beta_9\left(K_4-\cos^2 2\Phi\right)+\beta_{10}\left(K_4-\cos^2 2\Phi\right)^2+\beta_{11}(I_C-3)\left(K_4-\cos^2 2\Phi\right) \\ & +\beta_{12}(K_1-2)\left(K_4-\cos^2 2\Phi\right). \end{aligned} \tag{15.11}$$

In order that the intermediate configuration is unstressed, we require that $\beta_1-2\beta_2=Jp_0$, $\beta_4+2\beta_8=0$, and $\beta_7=\beta_9=0$. Klisch and Lotz (1999) used a strain energy representation (13.16) with Q defined by (15.10) to model compressible and orthotropic cartilaginous tissues. They also used a strain energy representation in terms of the sum of exponential functions, in which the dependence on the invariants is placed in the separate exponentials.

15.2. Evolution equations for stretch ratios

Since two sets of fibers are equivalent, we suppose that they grow equally if unrestricted by stress. Thus, their growth stretch ratio is equal and we have isotropic growth in the planes parallel to $\mathbf{a}^0$ and $\mathbf{b}^0$. Consequently, the overall growth is transversely isotropic with the axis of isotropy parallel to $\mathbf{a}^0\times\mathbf{b}^0$. Denoting the unit vector in this direction by $\mathbf{c}^0$, we have

$$\mathbf{F}_\mathrm{g}=\vartheta_\mathrm{g}\mathbf{I}+(\eta_\mathrm{g}-\vartheta_\mathrm{g})\mathbf{c}^0\otimes\mathbf{c}^0, \tag{15.12}$$

where η_g is the stretch ratio in the direction $\mathbf{c}^0$, and ϑ_g is the stretch ratio in any orthogonal direction. We suggest the following functions for their evolution equations

$$\dot{\eta}_\mathrm{g}=f_\eta\left[\eta_\mathrm{g},\vartheta_\mathrm{g},T_\mathrm{c}^\mathrm{e},\tfrac{1}{2}\left(\operatorname{tr}\mathbf{T}_\mathrm{e}-T_\mathrm{c}^\mathrm{e}\right)\right], \tag{15.13}$$

$$\dot{\vartheta}_\mathrm{g}=f_\vartheta\left\{\eta_\mathrm{g},\vartheta_\mathrm{g},\tfrac{1}{2}\left(T_\mathrm{a}^\mathrm{e}+T_\mathrm{b}^\mathrm{e}\right),\tfrac{1}{2}\left[\operatorname{tr}\mathbf{T}_\mathrm{e}-\tfrac{1}{2}\left(T_\mathrm{a}^\mathrm{e}+T_\mathrm{b}^\mathrm{e}\right)\right]\right\}. \tag{15.14}$$

The average normal stress in the fiber directions is

$$\tfrac{1}{2}\left(T_\mathrm{a}^\mathrm{e}+T_\mathrm{b}^\mathrm{e}\right)=\tfrac{1}{2}\left(\mathbf{a}^0\cdot\mathbf{T}_\mathrm{e}\cdot\mathbf{a}^0+\mathbf{b}^0\cdot\mathbf{T}_\mathrm{e}\cdot\mathbf{b}^0\right), \tag{15.15}$$

while

$$\tfrac{1}{2}\left[\operatorname{tr}\mathbf{T}_\mathrm{e}-\tfrac{1}{2}\left(T_\mathrm{a}^\mathrm{e}+T_\mathrm{b}^\mathrm{e}\right)\right] \tag{15.16}$$

represents the average normal stress in the planes perpendicular to the fibers. These two stresses are believed to have a dominant mechanical effect on the change of the stretch ratio ϑ_g. Likewise, the normal stress $T_\mathrm{c}^\mathrm{e}=\mathbf{c}^0\cdot\mathbf{T}_\mathrm{e}\cdot\mathbf{c}^0$ in the direction $\mathbf{c}^0$ and the average normal stress in the plane perpendicular to $\mathbf{c}^0$ are believed to have a dominant mechanical effect on the change of the stretch ratio η_g. In the simplest case of linear stress dependence, the evolution equations are

$$\dot{\eta}_\mathrm{g}=k_\eta(\eta_\mathrm{g})\left[T_\mathrm{c}^\mathrm{e}-v_\eta\left(\operatorname{tr}\mathbf{T}_\mathrm{e}-T_\mathrm{c}^\mathrm{e}\right)\right], \tag{15.17}$$

$$\dot{\zeta}_g = k_\zeta(\zeta_g)\left\{\tfrac{1}{2}\left(T^e_a + T^e_b\right) - v_\vartheta\left[\operatorname{tr}\mathbf{T}_e - \tfrac{1}{2}\left(T^e_a + T^e_b\right)T^e_n\right]\right\}, \tag{15.18}$$

where v_η and v_ϑ are the constants, or appropriate functions of η_g and ϑ_g. To prevent an unlimited growth at an arbitrary non-zero state of stress, we propose that during the mass growth k_η is specified by

$$k_\eta(\eta_g) = k^+_{\eta 0}\left(\frac{\eta^+_g - \eta_g}{\eta^+_g - 1}\right)^{m^+_\eta}, \quad T^e_c > \frac{v_\eta}{1+v_\eta}\operatorname{tr}\mathbf{T}_e. \tag{15.19}$$

The limiting value of the stretch ratio η_g is $\eta^+_g > 1$, and $k^+_{\eta 0}$ and m^+_η are the material parameters. In the case of the mass resorption, the corresponding expression is

$$k_\eta(\eta_g) = k^-_{\eta 0}\left(\frac{\eta_g - \eta^-_g}{1 - \eta^-_g}\right)^{m^-_\eta}, \quad T^e_c < \frac{v_\eta}{1+v_\eta}\operatorname{tr}\mathbf{T}_e. \tag{15.20}$$

Similarly, k_ϑ is specified by

$$k_\vartheta(\vartheta_g) = k^+_{\vartheta 0}\left(\frac{\vartheta^+_g - \vartheta_g}{\vartheta^+_g - 1}\right)^{m^+_\vartheta}, \quad \frac{1}{2}\left(T^e_a + T^e_b\right) > \frac{v_\vartheta}{1+v_\vartheta}\operatorname{tr}\mathbf{T}_e, \tag{15.21}$$

where $\vartheta^+_g > 1$ is the limiting value of the stretch ratio ϑ_g that can be attained by mass growth, and $k^+_{\vartheta 0}$ and m^+_ϑ are the material parameters. In the case of the mass resorption, the corresponding expression is

$$k_\vartheta(\vartheta_g) = k^-_{\vartheta 0}\left(\frac{\vartheta_g - \vartheta^-_g}{1 - \vartheta^-_g}\right)^{m^-_\vartheta}, \quad \frac{1}{2}\left(T^e_a + T^e_b\right) < \frac{v_\vartheta}{1+v_\vartheta}\operatorname{tr}\mathbf{T}_e. \tag{15.22}$$

16. Conclusions

We have presented in this paper a general constitutive theory of stress-modulated growth of biomaterials, with an accent given to pseudo-elastic soft tissues capable of large deformations, such as blood vessels and muscles. The governing equations of the mechanics of solids with a growing mass are first derived by extending the classical thermodynamic analysis of solids with a constant mass. The formulation is given in the framework of finite deformations. The resulting constitutive structures are given by equations such as (7.9)–(7.12). The multiplicative decomposition of the deformation gradient into its elastic and growth parts is subsequently used to cast the constitutive theory in an incremental rate-type form. The rate of deformation tensor is partitioned into elastic and growth parts. Isotropic, transversely isotropic, and two types of orthotropic biomaterials are considered. Specific forms of the elastic strain energy suitable for soft tissues are considered, and the corresponding instantaneous elastic moduli are derived. An explicit representation of the growth part of deformation gradient is constructed in each case; e.g., Eq. (14.13) for orthotropic mass growth. The stress-dependent evolution equations for the growth stretch ratios are proposed for isotropic, transversely isotropic, and orthotropic mass growth. For example, if an orthotropic tissue undergoes a transversely isotropic mass growth, these are given by Eqs. (15.17) and (15.18), assuming the linear stress dependence. The material parameters that appear in these expressions should be specified in accordance with the experimental data obtained for the particular tissue. This is clearly an essential aspect of future research. Appealing tests include those with a transmural radial cut through the blood vessel, which relieves the residual stresses due to differential growth of its inner and outer layers. The opening angle then provides a convenient measure of the circumferential residual strain, as discussed by Liu and Fung (1988, 1989), Fung (1993), Humphrey (1995), Taber and Eggers (1996), and others.

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Appendix A

The coefficients appearing in the expressions for the elastic moduli components $\Lambda^{\mathrm{e}}_{ijkl}$ of Eqs. (12.10), (13.9) and (14.8) are defined in terms of the gradients of elastic strain energy with respect to individual and mixed invariants of the elastic deformation and structural tensors as follows:

$$a_1 = \frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial I_C^2} - 2I_C\,\frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial I_C\,\partial II_C} + I_C^2\,\frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial II_C^2} - \frac{\partial(\rho_{\mathrm{g}}^0\psi)}{\partial II_C}, \tag{A.1}$$

$$a_2 = \frac{\partial(\rho_{\mathrm{g}}^0\psi)}{\partial II_C}, \tag{A.2}$$

$$a_3 = 2\left[\frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial I_C\,\partial II_C} - I_C\,\frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial II_C^2}\right], \tag{A.3}$$

$$a_4 = \frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial II_C^2}, \tag{A.4}$$

$$a_5 = 2\left[III_C\,\frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial III_C\,\partial I_C} - III_C I_C\,\frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial II_C\,\partial III_C}\right], \tag{A.5}$$

$$a_6 = 2\left[III_C\,\frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial II_C\,\partial III_C}\right], \tag{A.6}$$

$$a_7 = III_C^2\,\frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial III_C^2} + III_C\,\frac{\partial(\rho_{\mathrm{g}}^0\psi)}{\partial III_C}, \tag{A.7}$$

$$a_8 = -III_C\,\frac{\partial(\rho_{\mathrm{g}}^0\psi)}{\partial III_C}, \tag{A.8}$$

$$a_9 = \frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial M_1\,\partial I_C} - I_C\,\frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial M_1\,\partial II_C}, \tag{A.9}$$

$$a_{10} = 2\,\frac{\partial(\rho_{\mathrm{g}}^0\psi)}{\partial M_2}, \tag{A.10}$$

$$a_{11} = \frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial M_1^2}, \tag{A.11}$$

$$a_{12} = \frac{\partial^2(\rho_{\mathrm{g}}^0\psi)}{\partial M_1\,\partial II_C}, \tag{A.12}$$

$$a_{13} = III_C \frac{\partial^2(\rho_g^0\psi)}{\partial M_1\,\partial III_C}, \tag{A.13}$$

$$a_{14} = 2\left[\frac{\partial^2(\rho_g^0\psi)}{\partial M_2\,\partial I_C} - I_C \frac{\partial^2(\rho_g^0\psi)}{\partial M_2\,\partial II_C}\right], \tag{A.14}$$

$$a_{15} = 4\frac{\partial^2(\rho_g^0\psi)}{\partial M_1\,\partial M_2}, \tag{A.15}$$

$$a_{16} = 2\frac{\partial^2(\rho_g^0\psi)}{\partial M_2\,\partial II_C}, \tag{A.16}$$

$$a_{17} = 2III_C \frac{\partial^2(\rho_g^0\psi)}{\partial M_2\,\partial III_C}, \tag{A.17}$$

$$a_{18} = 2\frac{\partial^2(\rho_g^0\psi)}{\partial M_2^2}. \tag{A.18}$$

The coefficients a_{19} to a_{28} are determined from the expressions (A.9)–(A.18) by replacing there M_1 and M_2 with N_1 and N_2.

The gradients of the coefficients d_r with respect to $\mathbf{C}_e$ appearing in Eq. (15.7) are

$$\frac{\partial d_r}{\partial \mathbf{C}_e} = b_{r1}\mathbf{I} + b_{r2}\mathbf{C}_e + b_{r3}\mathbf{C}_e^{-1} + k_{r1}(\mathbf{a}^0\otimes\mathbf{a}^0 + \mathbf{b}^0\otimes\mathbf{b}^0) + k_{r2}\Big[\left(\mathbf{a}^0\otimes\mathbf{a}^0\right)\left(\mathbf{b}^0\cdot\mathbf{C}_e\cdot\mathbf{b}^0\right) + \left(\mathbf{b}^0\otimes\mathbf{b}^0\right)\left(\mathbf{a}^0\cdot\mathbf{C}_e\cdot\mathbf{a}^0\right)\Big] + k_{r3}\Big[\mathbf{a}^0\otimes\left(\mathbf{C}_e\cdot\mathbf{a}^0\right) + \left(\mathbf{C}_e\cdot\mathbf{a}^0\right)\otimes\mathbf{a}^0 + \mathbf{b}^0\otimes\left(\mathbf{C}_e\cdot\mathbf{b}^0\right) + \left(\mathbf{C}_e\cdot\mathbf{b}^0\right)\otimes\mathbf{b}^0\Big] + \tfrac{1}{2}k_{r4}\left(\mathbf{a}^0\otimes\mathbf{b}^0 + \mathbf{b}^0\otimes\mathbf{a}^0\right)\cos 2\Phi, \quad r = 1,2,3,4, \tag{A.19}$$

where

$$b_{r1} = \frac{\partial^2(\rho_g^0\psi)}{\partial I_C\,\partial K_r} - I_C \frac{\partial^2(\rho_g^0\psi)}{\partial II_C\,\partial K_r}, \tag{A.20}$$

$$b_{r2} = \frac{\partial^2(\rho_g^0\psi)}{\partial II_C\,\partial K_r}, \quad b_{r3} = III_C \frac{\partial^2(\rho_g^0\psi)}{\partial III_C\,\partial K_r}, \tag{A.21}$$

$$k_{rs} = \frac{\partial^2(\rho_g^0\psi)}{\partial K_r\,\partial K_s}, \quad r,s = 1,2,3,4. \tag{A.22}$$

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